

Non-Adiabatic Mixing of Two Bose-Einstein Condensates

von

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Ich versichere, dass ich die Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die Zitate kenntlich gemacht habe.

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Introduction

The idea of Bose-Einstein condensation (BEC) dates back to 1925 when A. Einstein, based on a work of S.N. Bose about statistical description of the quanta of light, predicted the occurrence of a phase transition in a gas of noninteracting atoms. This phase transition is connected with the condensation of bosonic atoms that occupy the lowest energy state. This occurs due to the quantum statistical effects.

It took about 70 years to verify Einsteins prediction. The experimental search for BEC started in the early 1970's, making use of techniques based on magnetic and optical trapping, and advanced cooling mechanisms. The first studies were focused on the spin polarized hydrogen. It was one of the most natural candidates for BEC because of its light mass. Experiments on hydrogen atoms, based on the techniques mentioned above, did not succeed due to the high rate of recombination of the individual atoms to molecules (Silvera [3] and Walraven [4], 1980 and 1986). Later, because of the advances in laser-based techniques, such as laser cooling and magneto-optical trapping, it was possible to cool alkali atoms to very low temperatures because of their favorable internal energy-level structure. Once the gas is trapped, the temperature can be lowered further by evaporative cooling (Ketterle and van Drute [2]). However, the equilibrium configuration of the system produced in such a way could be a solid phase. In order to observe BEC, one has therefore to maintain the system in a metastable gas phase for sufficient time. Moreover, the system should be sufficiently dilute, so that the three-body collisions, which are responsible for a solid phase, can be neglected. All these difficulties were successfully overcome by Cornell, Wieman and Ketterle [2, 1]. They observed BEC in a trapped system of Rubidium atoms cooled down to 170 nK. For this work they received the Nobel prize in 2001.

The technique used by Cornell and Wieman at Boulder and Ketterle at MIT is based on the experimental studies of dilute atomic gases. They combined different cooling techniques and succeeded in reaching the temperatures and the densities required to observe BEC.

From the beginning the experimental and the theoretical research on this unique phenomenon predicted by quantum statistics has involved different areas of physics. F. London, directly after the discovery of superfluidity in liquid Helium (^4He), postulated that it could be a manifestation of Bose-Einstein condensation. Landau developed the first self-consistent theory of superfluids in terms of the spectrum of

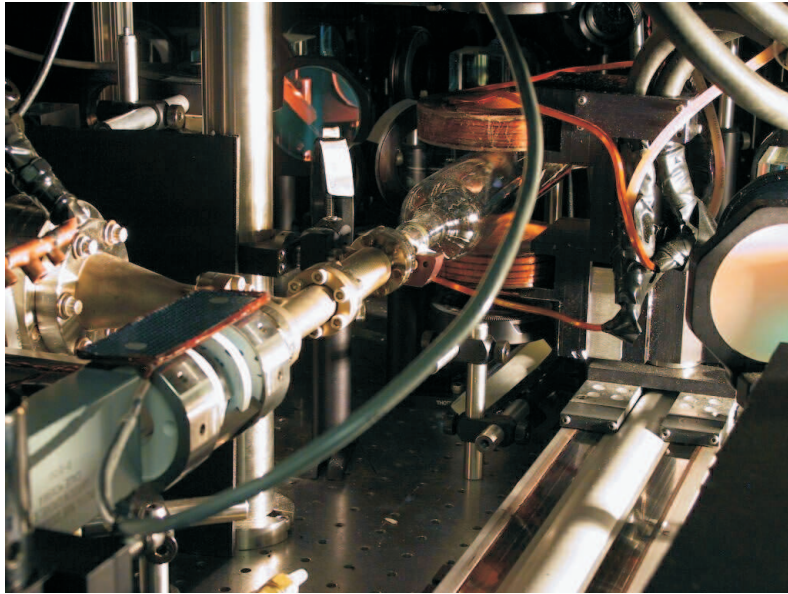


Figure 1: "Easy" BEC machine. From [<http://jilawww.colorado.edu/bec/>]

elementary excitations of the fluid. This two-fluid hydrodynamics was proven to be related to the thermodynamic Green's function formulation of the many-body problem [36]. Evidence for BEC in Helium emerged later from the analysis of the momentum distribution of the atoms measured in neutron-scattering experiments. In recent years, BEC has been investigated in the gas of paraexcitons in semiconductors, but up to now there is no evidence for it.

With the possibility of cooling down bosons, the question arises if it is whether or not realizable for fermions. Recently experimentalists managed to cold down fermions in spite of the Pauli exclusion principle at a very low temperature, which forbids that two fermions occupy the same quantum state. To exhibit Bose-Einstein condensation, the fermions must form pairs, i.e. compound particles (e.g. molecules) which are bosons. In order to achieve it experimentally, one makes use of the Feshbach resonance for changing the coupling strength between the fermions (BCS-BEC crossover). The first molecular Bose-Einstein condensates were created in November 2003 by the groups of Rudolf Grimm at the University of Innsbruck, Deborah S. Jin at the University of Colorado at Boulder and Wolfgang Ketterle at MIT [6, 7, 8].

One of the most important features of the experimentally realized trapped Bose gases is that they are inhomogeneous and finite systems, with the number of particles ranging from few thousands to millions. The inhomogeneity of alkali BEC is due to an external potential, which is used to confine atoms in real space. As a result, Bose-Einstein condensation shows up in coordinate space (see Fig. 2), making the direct experimental investigation of the condensation feasible and providing new possibilities for interesting studies from both experimental and theoretical

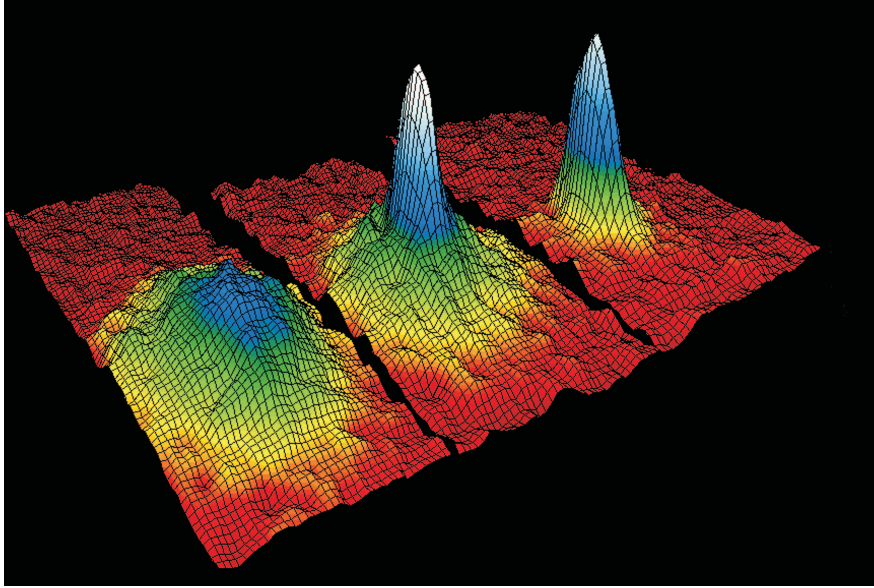


Figure 2: Images of the velocity distribution of Rubidium atoms. The left image corresponds to the gas at a temperature above the critical temperature where the atoms can occupy the energy levels according to Bose-Einstein statistics. The center frame shows the appearing condensate (condensation state), i.e. the occupation of the lowest state of energy by the atoms. And the right frame shows the condensate after further evaporation when the majority of atoms are still in a condensed state. From Cornell (1996)

point of view. It also allows to study in a controlled way new physical quantities and phenomena which were not accessible previously: for example, the temperature dependence of the condensate, energy and density distribution, interference phenomena, etc.

Motivation and Goal

In this thesis we consider a system consisting of two independent Bose-Einstein condensates confined in a double well potential. The behavior of the system when the two condensates are connected presents a fundamental problem in non-equilibrium macroscopic quantum mechanics. The two condensates can be described by an eigenstate $|z\rangle$, referring to the interwell population imbalance, which can be written in the phase representation as follows

$$|z\rangle = \int_0^{2\pi} \frac{d\theta}{\sqrt{2\pi}} e^{iz\theta} |\theta\rangle,$$

where $|\theta\rangle$ is an eigenstate of the relative phase difference between the condensates. The state may be understood as a coherent superposition of all possible relative phases. As far as the condensates are separated, e.g., by a barrier that is sufficiently high and/or broad, so that the particles cannot tunnel from one condensate to the other, the system presents no problems. The situation changes when the condensates are brought together, for example, by lowering and/or narrowing the barrier separating the condensates. In this way a Josephson-like tunneling is induced. As a consequence, decoherence sets in, and the relative phase can be effectively measured.¹ In the present work we assume that dissipation is caused by creation of Bogoliubov quasiparticles above the barrier, which can contribute to the particle flow from one well to the other.

The problem under consideration in this thesis presents two relevant aspects: First, it presents an important case of quantum measurement process that can be treated in detail. Second, the connection between independent Bose-Einstein condensates has features of a macroscopic interference experiment [13, 15, 14], since it involves an initial random relative phase which becomes well defined.

Previous studies on Josephson tunneling between two condensates are based on the two-mode approximation [27, 29] or on the two-site Bose-Hubbard model [31, 28]. In the two-mode approximation one works in the low-energy regime, which is governed by the Gross-Pitaevskii equation for the condensate wave function.

¹In general the system undergoes to a state far from equilibrium, but due to the interactions with the dissipative environment, it eventually relaxes to an equilibrium state.

The dynamics of the system is well explained in terms of the interwell population imbalance and the relative phase. It can be shown also that equations of motion of the system can be mapped to equations describing a mathematical pendulum. Numerical analysis of these equations show two different regimes depending on the initial population imbalance and on the self-interaction parameter. The theoretical results obtained in [27] were corroborated experimentally. Markus Oberthaler and his research group realized a single Bose-Josephson junction [15]. They first created a Bose condensate in a 3D-harmonic trap. Then they superposed it with a 1D-periodical trap with very large lattice spacing. In this way they split the initial condensate into two components. In the new configuration of the system they measured the population imbalance as a function of time. Then, at a certain time $t = t_0$ they switched off the 1D trap and observed the interference pattern, from which they extracted the relative phase. They repeated the procedure for different times t_0 in order to obtain the relative phase as a function of time. On the other side the studies on the two-site Bose-Hubbard model presented the quantum dynamics of the system (neglecting dissipation). The calculations presented in [31, 28] are limited to a small number of particles, which not correspond to the experimental realizations. This two different approaches do not include non-equilibrium features appearing in the experiments. Thus, they have to be extended and this is the aim of this thesis.

An additional motivation to study the mixing of condensates is the atom laser. Similarities between Bose-Einstein condensation and laser light suggest the experimental realization of a bright, coherent matter wave, which will open new avenues of fundamental tests of quantum mechanics. The possible creation of a continuous, coherent atomic beam, has been a topic of great interest. One of the major challenges lies in the obtention of continuous condensation of atomic gases due to the stringent cooling conditions. Alternatively, one can realize a continuous source of condensate either by bringing new condensates into the trap and uniting them [17], or by coupling to spatially separated condensates with large particle number [16]. If the mixing of condensates takes place in a non adiabatic way, new features arise, for example, concerning the strength of the depletion of the condensates. It is therefore interesting to study the elementary excitations, because they have the same spectrum as the density fluctuations [39].

In the present work we consider a system consisting of two Bose-Einstein condensates with large particle numbers confined in a double well potential at zero temperature. Upon sudden lowering of the barrier between the condensates, the condensates mix and Bogoliubov quasiparticles above the condensates are created, absorbing part of the system's entropy. The non-adiabatic switching process is described by projecting the exact Hamiltonian onto a sub-Hilbertspace comprised of a two-mode approximation for the condensates and single-particle excitations. We use non-equilibrium field theoretical methods to compute the kinetic equations for the non-condensate particles and the associated equations for the two condensates

wave functions.

In the following chapter 1 we introduce the field theoretical methods for bosons in the presence of a condensate. Since the subject of study is a system consisting of bosons at zero temperature that is suddenly perturbed, it displays non-equilibrium features which require the non-equilibrium extension presented in Chapter 2. This completes the theoretical tools needed to study the problem. In Chapter 4 we present our model, and perform the microscopic derivation of the Hamiltonian. We compute its corresponding kinetic equations for the distribution functions, and the associated equation of motion for the two macroscopic condensate wave functions. In order to have a better understanding how the inclusion of the single-particle excitation (which have the same spectrum as the condensate amplitude oscillations) affects the system, we compare the dynamics of system described with the non-equilibrium theoretical tools with the mean field approximation presented in Chapter 3. In Chapter 5 we present our numerical results for the equations computed within the Bogoliubov-Hartree-Fock approximation and discuss our finding.

Chapter 1

Field Theory for a System of Bosons at Zero Temperature

In this chapter we consider a system of weakly interacting bosons at zero temperature in the presence of external trapping potential. Most of the bosons in this case are concentrated in the lowest state of energy. As a consequence, averages of normal products of Bose creation and annihilation operators are not only nonvanishing, but can actually be arbitrarily large, and this should be taken into account while constructing a field theoretical approach. Therefore, the generalization of the methods of quantum field theory to the case of a system of interacting bosons at temperatures below the temperature of Bose-Einstein condensation contains large difficulties. Nevertheless, the appropriate formalism has been developed by S. T. Beliaev [20]. In this chapter we present the path integral approach, followed by the derivation of the Dyson-Beliaev equations for the Green's function [24, 34].

We will first show how the mean field Gross-Pitaevskii comes about and then proceed to the inclusion of the Hartree-Fock corrections to the non-condensate particle Green's function and derive the Bogoliubov-Hartree-Fock coupled equations.

We consider the general Hamiltonian of a system of interacting bosons

$$H = \int d^3x \Psi^\dagger(\vec{x}, t) \left(-\frac{1}{2m} \Delta + V_{\text{ext}}(\vec{x}, t) - \mu \right) \Psi(\vec{x}, t) + \frac{1}{2} \int d^3x \int d^3y \Psi^\dagger(x, t) \Psi^\dagger(y, t) U(x - y) \Psi(y, t) \Psi(x, t) \quad (1.0.1)$$

Here are $\Psi^\dagger(\vec{x}, t)$ and $\Psi(\vec{x}, t)$ the creation and annihilation field operators, respectively. V_{ext} is the trapping potential and $U(\vec{x} - \vec{y})$ is the two body interaction potential.

Since Bose dilute gases are considered throughout of this thesis, the interaction between the bosons is local, i.e.

$$U(\vec{x} - \vec{y}) = g\delta(\vec{x} - \vec{y}), \quad (1.0.2)$$

and the Hamiltonian (1.0.1) is reduced to

$$\begin{aligned}
H &= \int d^3x \Psi^\dagger(\vec{x}, t) \left(-\frac{1}{2m} \Delta + V_{\text{ext}}(\vec{x}, t) - \mu \right) \Psi(\vec{x}, t) \\
&\quad + \frac{g}{2} \int d^3x \Psi^\dagger(\vec{x}, t) \Psi^\dagger(\vec{x}, t) \Psi(\vec{x}, t) \Psi(\vec{x}, t),
\end{aligned} \tag{1.0.3}$$

where

$$g = \frac{4\pi a_s}{m} \tag{1.0.4}$$

and a_s as the s -wave scattering length.

The partition function of this model can be written as a path integral

$$Z = \int d[\phi^*] d[\phi] e^{iS[\phi^*, \phi]} \tag{1.0.5}$$

with the action S being a functional of the fields ϕ^* and ϕ corresponding to the operators Ψ^\dagger and Ψ , respectively:

$$\begin{aligned}
S[\phi^*, \phi] &= \int dt \int d^3x \phi^*(\vec{x}, t) \left(i \frac{\partial}{\partial t} - \left[-\frac{1}{2m} \Delta + V_{\text{ext}}(\vec{x}, t) - \mu \right] \right) \phi(\vec{x}, t) \\
&\quad - \frac{g}{2} \int dt \int d^3x \phi^*(\vec{x}, t) \phi^*(\vec{x}, t) \phi(\vec{x}, t) \phi(\vec{x}, t)
\end{aligned} \tag{1.0.6}$$

Now that we got the action describing our system we will present first in the proceeding section the mean field approach and derive the *Gross-Pitaevskii equation* for the condensate wave function. Then we will compute the corrections to this result in a systematic way.

1.1 Condensate Order Parameter and Gross-Pitaevskii Equation

In a system consisting of bosons at zero temperature the phase transition of interest is the Bose-Einstein condensation. The associated order parameter is defined

$$\phi_0(\vec{x}, t) = \langle \phi(\vec{x}, t) \rangle \tag{1.1.1}$$

which corresponds to the condensate wave function. We notice that below the critical temperature the order parameter is finite. Above the critical temperature its contribution to the field is negligibly small.

Consider now the system at zero temperature and assume that all particles are in the ground state. It is therefore an appropriate approximation to replace the fields

ϕ^* and ϕ by their expectations values $\langle \phi^* \rangle$ and $\langle \phi \rangle$, respectively. Thus the action takes the form

$$S[\phi_0^*, \phi_0] = \int dt \int d^3x \phi_0^*(\vec{x}, t) \left(i \frac{\partial}{\partial t} - \left[-\frac{1}{2m} \Delta + V_{\text{ext}}(\vec{x}, t) - \mu \right] \right) \phi_0(\vec{x}, t) - \frac{g}{2} \int dt \int d^3x \phi_0^*(\vec{x}, t) \phi_0^*(\vec{x}, t) \phi_0(\vec{x}, t) \phi_0(\vec{x}, t), \quad (1.1.2)$$

where the action S is now a functional of ϕ_0^* and ϕ_0 .

In order to derive the equation of motion for the condensate wave function ϕ_0 we variate the action (1.1.2) with respect to ϕ_0^* and require this variation (saddle point approximation) to be zero. We immediately obtain

$$\left(i \frac{\partial}{\partial t} - \left[\frac{-1}{2m} \Delta + V_{\text{ext}}(\vec{x}, t) + g |\phi_0(\vec{x}, t)|^2 - \mu \right] \right) \phi_0(\vec{x}, t) = 0 \quad (1.1.3)$$

This equation is called *Gross-Pitaevskii equation*. It was derived independently by Gross and Pitaevskii and is one of the most frequently used theoretical tools for investigating non-uniform dilute Bose gases at low temperatures. It has the form of a mean field equation where the order parameter must be calculated in a self-consistent way.

Now let us turn our attention to the order parameter. We observe that the order parameter is normalized to the total number of particles

$$N = \int d^3x |\phi_0(\vec{x}, t)|^2. \quad (1.1.4)$$

This implies that

$$n(\vec{x}, t) = |\phi_0(\vec{x}, t)|^2 \quad (1.1.5)$$

where $n(\vec{x}, t)$ is the density of the condensate, which coincides with the density of the gas in this approximation. Moreover one can state, based on Eq. (1.1.5), that the order parameter may be written as

$$\phi_0(\vec{x}, t) = \sqrt{n(\vec{x}, t)} e^{i\theta(\vec{x}, t)} \quad (1.1.6)$$

where $\theta(\vec{x}, t)$ is the phase of the condensate wave function. It also can be shown that ϕ_0 satisfies the hydrodynamic equations

$$0 = \frac{\partial}{\partial t} n(\vec{x}, t) + \text{div} \left(n(\vec{x}, t) \left(\frac{1}{m} \nabla \theta(\vec{x}, t) \right) \right) \quad (1.1.7)$$

$$0 = \frac{\partial}{\partial t} \theta(\vec{x}, t) - \mu + \frac{1}{2} m v_\theta^2 + V_{\text{ext}} + g n(\vec{x}, t) - \frac{1}{2m \sqrt{n(\vec{x}, t)}} \Delta \sqrt{n(\vec{x}, t)} \quad (1.1.8)$$

with $v_\theta = \nabla\theta$ as the fluid velocity and the phase $\theta(\vec{x}, t)$ as its potential.¹ Moreover we can prove with the help of the hydrodynamic equations that the density $n(\vec{x}, t)$ and the phase $\theta(\vec{x}, t)$ are canonical conjugated, i.e.

$$\hat{n}(\vec{x}, t)\hat{\theta}(\vec{y}, t) - \hat{\theta}(\vec{y}, t)\hat{n}(\vec{x}, t) = i\delta(\vec{x} - \vec{y}). \quad (1.1.9)$$

Here we used the "hat" in order to differentiate between the operators \hat{n} and $\hat{\theta}$ after the quantization from the dynamical variables n and θ .

1.2 Bogoliubov Approximation

The standard field theoretical approach makes use of the fact that the average of a product of several noninteracting operators Ψ and Ψ^\dagger can be reduced to products of averages of pairs of the operators. This is a consequence of Wick's theorem, which states that the average of a chronological product of any number of field operators decomposes into a sum of normal products with all possible pairings. For a system of fermions, the ground state or "vacuum" is such that the normal products can be made to vanish by properly defining the annihilation and creation operators. In fact, in a Bose gas at low temperatures, an arbitrarily huge number of particles can be "condensed" in the lowest state of energy. Thus, a characteristic feature of the state of Bose-Einstein condensation is that the density of particles in the lowest state of energy approaches a finite value in the thermodynamic limit,² i.e.

$$n_0(\vec{x}, t) \xrightarrow[V, N \rightarrow \infty]{\frac{N}{V} = \text{const}} \text{finite} \quad (1.2.1)$$

where n_0 is the particle density in the lowest state of energy, N the total number of particles, and V the volume of the system. From the preceding section we know that the number density of particles in the lowest state of energy is connected to the order parameter. Therefore, a finite n_0 means a finite order parameter $\phi_0 = \langle \Psi \rangle$. This has as a consequence that the averages of normal products of bosonic field operators are not only nonvanishing, but can be arbitrarily large.

In order to construct a field theory for a bosonic system at zero temperature, we have to do an appropriate approximation [18, 20]. This theory will be an extension of the mean field theory presented in the preceding section, which will determine the corrections to Gross-Pitaevskii equation, derived in section 1.1.

We consider first the action (1.0.6), which is associated to a dilute gas of spinless

¹Such a definition of the fluid velocity and its potential is only valid for rotationless fluids.

²The limit presented here is well defined for uniform systems. In a trapped gas the density is not uniform and consequently the thermodynamic limit should be defined in a different way. For more details see [11]

bosons at zero temperature

$$\begin{aligned}
S[\phi^*, \phi] &= \int dt \int d^3x \phi^*(\vec{x}, t) \left(i \frac{\partial}{\partial t} - \left[-\frac{1}{2m} \Delta + V_{\text{ext}}(\vec{x}, t) - \mu \right] \right) \phi(\vec{x}, t) \\
&\quad - \frac{g}{2} \int dt \int d^3x \phi^*(\vec{x}, t) \phi^*(\vec{x}, t) \phi(\vec{x}, t) \phi(\vec{x}, t).
\end{aligned} \tag{1.2.2}$$

Then we expand ϕ around the order parameter

$$\phi(\vec{x}, t) = \phi_0(\vec{x}, t) + \phi'(\vec{x}, t) \tag{1.2.3}$$

where ϕ' is the field corresponding to the particles out the condensate. It is important to realize that in order to consistently define the fluctuations ϕ' in this manner, we have to require that

$$\int d^3x \phi_0^*(\vec{x}, t) \phi'(\vec{x}, t) = \int d^3x \phi_0(\vec{x}, t) (\phi')^*(\vec{x}, t) = 0. \tag{1.2.4}$$

The physical reason for this condition is that ϕ' should contain all the configurations that are orthogonal to ϕ_0 . We find after the above substitution that

$$S[\phi^*, \phi] = S[\phi_0^*, \phi_0] + S_0[\phi^*, \phi] + S_{\text{int}}[\phi^*, \phi], \tag{1.2.5}$$

where the linear and quadratic terms in ϕ' are given by

$$\begin{aligned}
S_0[\phi^*, \phi] &= \int dt \int d^3x (\phi')^* \left(i \frac{\partial}{\partial t} - \left[-\frac{1}{2m} \Delta + V_{\text{ext}}(\vec{x}, t) + g|\phi_0|^2 - \mu \right] \right) \phi_0 \\
&+ \int dt \int d^3x \phi_0^* \left(i \frac{\partial}{\partial t} - \left[-\frac{1}{2m} \Delta + V_{\text{ext}}(\vec{x}, t) + g|\phi_0|^2 - \mu \right] \right) \phi' \\
&+ \int dt \int d^3x (\phi')^* \left(i \frac{\partial}{\partial t} - \left[-\frac{1}{2m} \Delta + V_{\text{ext}}(\vec{x}, t) + 2g|\phi_0|^2 - \mu \right] \right) \phi' \\
&- \frac{g}{2} \int dt \int d^3x (\phi_0)^2 (\phi')^* (\phi')^* \\
&- \frac{g}{2} \int dt \int d^3x (\phi_0^*)^2 \phi' \phi'
\end{aligned} \tag{1.2.6}$$

and the cubic and quartic terms by

$$\begin{aligned}
S_{\text{int}}[\phi^*, \phi] &= -g \int dt \int d^3x \phi_0 \phi' (\phi')^* (\phi')^* \\
&- g \int dt \int d^3x \phi_0^* (\phi')^* \phi' \phi' \\
&- \frac{g}{2} \int dt \int d^3x (\phi')^* (\phi')^* \phi' \phi',
\end{aligned} \tag{1.2.7}$$

where the fields ϕ_0 and ϕ' are both time and spatial dependent.

In the *Bogoliubov approximation* we neglect the last three interaction terms [18, 11]. Furthermore we observe that the linear terms in ϕ' and $(\phi')^*$ vanish, according to the Gross-Pitaevskii equation. In this manner it is ensured that

$$\langle \phi(\vec{x}, t) \rangle = \phi_0(\vec{x}, t) \quad \text{and} \quad \langle \phi'(\vec{x}, t) \rangle = 0. \quad (1.2.8)$$

Moreover, we see that the total density of the gas splits within this approximation into two components

$$\begin{aligned} n(\vec{x}, t) &= \langle \phi^*(\vec{x}, t+) \phi(\vec{x}, t) \rangle \\ &= |\phi_0(\vec{x}, t)|^2 + \langle (\phi')^*(\vec{x}, t+) \phi'(\vec{x}, t) \rangle \\ &= n_0(\vec{x}, t) + n'(\vec{x}, t) \end{aligned} \quad (1.2.9)$$

where $n_0(\vec{x}, t)$ and $n'(\vec{x}, t)$ are the densities of the particles in the condensate and above the condensate, respectively. The notation $t+$ means that the time argument of $\phi^*(\vec{x}, t+)$ is infinitesimally larger than the time argument of $\phi(\vec{x}, t)$.

The total number of particles in the condensate is given by Eq. (1.1.4). As it is displayed by Eq. (1.1.5), the number of particles in the condensate is in general smaller than the total number of particles in the gas due to the effect of the fluctuations.

Now we assume that the Gross-Pitaevskii equation (Eq. (1.1.3)) has already been solved. In order to determine the corrections in the Bogoliubov approximation we consider the quadratic term in the action

$$\begin{aligned} S_0[\phi^*, \phi] &= \int dt \int d^3x (\phi')^* \left(i \frac{\partial}{\partial t} - \left[-\frac{1}{2m} \Delta + V_{\text{ext}}(\vec{x}, t) + 2g|\phi_0|^2 - \mu \right] \right) \phi' \\ &\quad - \frac{g}{2} \int dt \int d^3x (\phi_0)^2 (\phi')^* (\phi') - \frac{g}{2} \int dt \int d^3x (\phi_0^*)^2 \phi' \phi' \\ &= \frac{1}{2} \int dt' \int dt \int d^3x' \int d^3x [(\phi')^*(\vec{x}, t), \phi'(\vec{x}, t)] \cdot \\ &\quad \cdot \begin{pmatrix} G_0^{-1}(\vec{x}, t; \vec{x}', t') & 0 \\ 0 & (G_0^{-1})^*(\vec{x}, t; \vec{x}', t') \end{pmatrix} \begin{bmatrix} (\phi')^*(\vec{x}', t') \\ \phi'(\vec{x}', t') \end{bmatrix} \\ &\quad - \frac{1}{2} \int dt' \int dt \int d^3x' \int d^3x [(\phi')^*(\vec{x}, t), \phi'(\vec{x}, t)] \cdot \\ &\quad \cdot \begin{pmatrix} 2g|\phi_0(\vec{x}, t)|^2 & g(\phi_0(\vec{x}, t))^2 \\ g(\phi_0^*(\vec{x}, t))^2 & 2g|\phi_0(\vec{x}, t)|^2 \end{pmatrix} \delta(t-t') \delta(\vec{x}, \vec{x}') \begin{bmatrix} (\phi')^*(\vec{x}', t') \\ \phi'(\vec{x}', t') \end{bmatrix} \\ &= \frac{1}{2} \int dt' \int dt \int d^3x' \int d^3x [(\phi')^*(\vec{x}, t), \phi'(\vec{x}, t)] \cdot \\ &\quad \cdot \mathbf{G}^{-1}(\vec{x}, t; \vec{x}', t') \begin{bmatrix} (\phi')^*(\vec{x}', t') \\ \phi'(\vec{x}', t') \end{bmatrix} \end{aligned} \quad (1.2.10)$$

with the noninteracting Green's function G_0 defined by

$$G_0^{-1}(\vec{x}, t; \vec{x}', t') = \left(i \frac{\partial}{\partial t} + \frac{1}{2m} \Delta_x - V_{\text{ext}} + \mu \right) \delta(t - t') \delta(\vec{x}, \vec{x}'), \quad (1.2.11)$$

and \mathbf{G} being the exterior product of $[(\phi')^*(\vec{x}, t), \phi'(\vec{x}, t)]$ and its hermitian conjugate. \mathbf{G} is the associated Green's function and has now a matrix structure, because not only the normal average $\langle (\phi')^*(\vec{x}, t) \phi'(\vec{x}', t') \rangle$ but also the anomalous average $\langle \phi'(\vec{x}, t) \phi'(\vec{x}, t) \rangle$ is now unequal to zero. This is a consequence of the finite value of the order parameter for temperatures below the critical temperature. Thus we have that

$$\mathbf{G}(\vec{x}, t; \vec{x}', t') = -i \begin{pmatrix} \langle \phi'(\vec{x}, t) (\phi')^*(\vec{x}', t') \rangle & \langle \phi'(\vec{x}, t) \phi'(\vec{x}', t') \rangle \\ \langle (\phi')^*(\vec{x}, t) (\phi')^*(\vec{x}', t') \rangle & \langle (\phi')^*(\vec{x}, t) \phi'(\vec{x}', t') \rangle \end{pmatrix}. \quad (1.2.12)$$

It is clear from Eq. (1.2.10) that \mathbf{G} satisfies the equation

$$\begin{aligned} \mathbf{G}^{-1}(\vec{x}, t; \vec{x}', t') &= \begin{pmatrix} G_0^{-1}(\vec{x}, t; \vec{x}', t') & 0 \\ 0 & (G_0^{-1})^*(\vec{x}, t; \vec{x}', t') \end{pmatrix} \\ &\quad - \begin{pmatrix} 2g|\phi_0(\vec{x}, t)|^2 & g(\phi_0(\vec{x}, t))^2 \\ g(\phi_0^*(\vec{x}, t))^2 & 2g|\phi_0(\vec{x}, t)|^2 \end{pmatrix} \delta(t - t') \delta(\vec{x}, \vec{x}'), \end{aligned} \quad (1.2.13)$$

which is clearly the lowest order result for the Green's function, because a perturbative treatment of S_{int} leads to higher order corrections. In general, the equation (1.2.13), known as the *Dyson-Beliaev equation*, can be written in the following form

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}^{-1} = \begin{pmatrix} G_0^{-1} & 0 \\ 0 & (G_0^{-1})^* \end{pmatrix} - \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{11} \end{pmatrix} \quad (1.2.14)$$

where G_{ij} for $i, j = 1, 2$ are the components of \mathbf{G} in the Bogoliubov space. Σ_{kl} for $k, l = 1, 2$ are the components for the selfenergy Σ . The off-diagonal elements are again called anomalous, since they vanish in the normal phase of the gas. Diagrammatically the Dyson-Beliaev equations are shown in Fig. 1.1. The self-energy in the Bogoliubov approximation is

$$\Sigma(\vec{x}, t; \vec{x}', t') = \begin{pmatrix} 2g|\phi_0(\vec{x}, t)|^2 & g(\phi_0(\vec{x}, t))^2 \\ g(\phi_0^*(\vec{x}, t))^2 & 2g|\phi_0(\vec{x}, t)|^2 \end{pmatrix} \delta(t - t') \delta(\vec{x}, \vec{x}'). \quad (1.2.15)$$

The diagrams corresponding to the different components of Σ are displayed in Fig. 1.2.

The Bogoliubov approach is only valid if the fluctuations are sufficiently small, since we have completely neglected the term in the action containing 3 and 4 fields

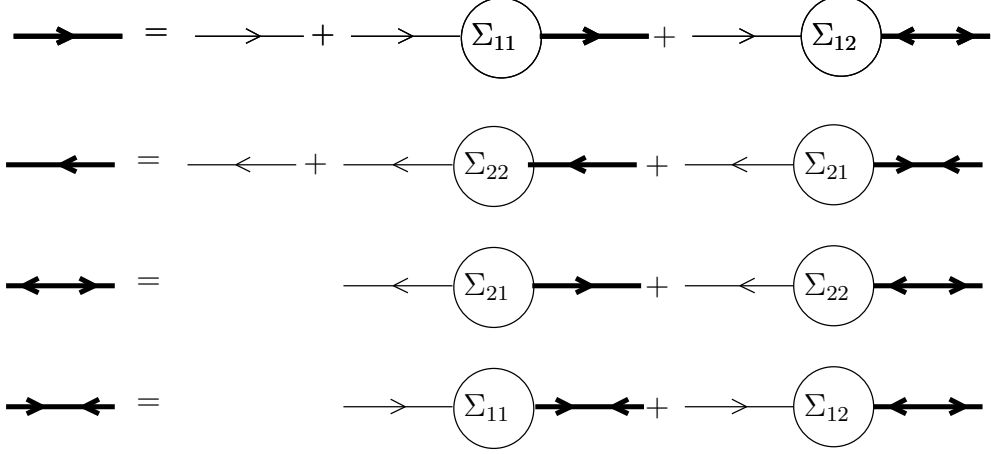


Figure 1.1: Exact Dyson-Belavev equations for the interacting normal and anomalous Green's functions.

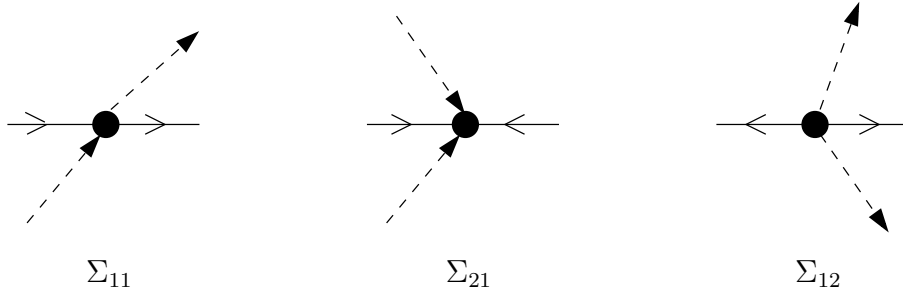


Figure 1.2: Bogoliubov self-energies. The dashed lines corresponds to condensate particles and the solid lines to the particles above the condensate.

ϕ' . Physically, this implies that the depletion of the condensate must be small. This approach can not therefore be applied to liquid helium, but it is expected to be valid for a weakly interacting atomic gas at sufficiently low temperatures. Under these conditions the Bogoliubov theory not only predicts the condensate density

$$n(\vec{x}, t) = |\phi_0(\vec{x}, t)|^2, \quad (1.2.16)$$

but also the collective modes of the condensate. Theoretically, the eigenfrequencies of the collective modes are again determined by the poles of the one particle Green's function for the particles above the condensate. In order to understand more clearly how these can be determined, we first consider a homogeneous Bose gas [18, 20, 11].

Consider now a uniform gas confined in a box of volume $V = L^3$. The one particle states are easily characterized by the wavevector

$$\vec{k} = \frac{2\pi\vec{n}}{L} \quad \text{for } \vec{n} \in \mathbb{Z}^3. \quad (1.2.17)$$

The one particle states and their corresponding energies are thus $e^{-i\epsilon_{\vec{k}}t+i\vec{k}\vec{x}}/\sqrt{V}$ and $\epsilon_{\vec{k}} = \frac{|\vec{k}|^2}{2m}$, respectively. Now making use of Fourier transformation we find for Eq. (1.2.13):

$$\mathbf{G}^{-1}(\vec{k}, \omega) = \begin{pmatrix} \omega - \epsilon_{\vec{k}} - g|\phi_0(\vec{x}, t)|^2 & g(\phi_0(\vec{x}, t))^2 \\ g(\phi_0^*(\vec{x}, t))^2 & -\omega - \epsilon_{\vec{k}} - g|\phi_0(\vec{x}, t)|^2 \end{pmatrix}. \quad (1.2.18)$$

Here we used the fact that the Gross-Pitaevskii equation reduces to

$$\mu = g|\phi_0|^2. \quad (1.2.19)$$

In this case the macroscopic wave function is independent of the position in the box. Moreover it is clear that the poles of \mathbf{G} are determined by its determinant. The poles deliver us the excitation spectrum of the system:

$$\omega_{\vec{k}} = \sqrt{\epsilon_{\vec{k}}^2 + 2g|\phi_0|^2\epsilon_{\vec{k}}} = \sqrt{\epsilon_{\vec{k}}^2 + 2gn_0\epsilon_{\vec{k}}}. \quad (1.2.20)$$

This is the famous *Bogoliubov dispersion relation* of the collective excitations.

For small momenta $|\vec{k}| \ll m|\vec{c}|$ the dispersion law of the quasi-particle takes a phonon-like form

$$\omega_{\vec{k}} = c\vec{k}, \quad (1.2.21)$$

where $\vec{c} = \sqrt{gn_0/m}$ is the sound velocity. This means that the Bogoliubov approximation predicts that the long wave excitations of a weakly interacting Bose gas are sound waves and can be regarded as the Goldstone modes associated with the breaking of gauge symmetry caused by Bose-Einstein condensation.

In the opposite limit $|\vec{k}| \gg m|\vec{c}|$ the dispersion law approaches the free particle law:

$$\omega_{\vec{k}} = \frac{\vec{k}^2}{2m} + gn_0. \quad (1.2.22)$$

The generalization to the inhomogeneous case is straightforward. In the same way as above, we first have to solve the Gross-Pitaevskii equation and fix the chemical potential. Then we can calculate the poles of \mathbf{G} , or equivalently and more conveniently, the zero's of \mathbf{G}^{-1} . Then the calculations reduces to the eigenvalue problem

$$\begin{pmatrix} h_0 + 2g|\phi_0|^2 & g(\phi_0)^2 \\ g(\phi_0^*)^2 & h_0 + 2g|\phi_0|^2 \end{pmatrix} \cdot \begin{pmatrix} u_n(\vec{x}) \\ v_n(\vec{x}) \end{pmatrix} = \begin{pmatrix} \omega_n & 0 \\ 0 & -\omega_n \end{pmatrix} \cdot \begin{pmatrix} u_n(\vec{x}) \\ v_n(\vec{x}) \end{pmatrix} \quad (1.2.23)$$

where we introduced the one-particle operator $h_0 = -\Delta/2m + V_{\text{ext}}(\vec{x}) - \mu$. This is called the *Bogoliubov-de Gennes equation*. Moreover, it can be shown that the functions u_n and v_n satisfy the orthonormalization relation

$$\int d^3x [u_i^*(\vec{x})u_j(\vec{x}) - v_i^*(\vec{x})v_j(\vec{x})] = \delta_{ij}. \quad (1.2.24)$$

Physically, the zero frequency solution $n = 0$ describes the dynamics of the global phase of the condensate. Because of the $U(1)$ symmetry of the action, this solution is essentially of no importance for the thermodynamic properties of a macroscopic gas system and is therefore usually neglected.

1.3 Bogoliubov-Hartree-Fock coupled equation

At temperatures near absolute zero, the Bogoliubov approximation is well defined. However at higher temperatures the number density of thermally excited particles $n'(\vec{x}, t) = iG_{11}(\vec{x}, t; \vec{x}, t+)$ becomes finite. We therefore treat the non-condensate part of the gas in the *Hartree-Fock approximation*, which is the lowest order approximation. Thus we find for the interaction part of the action

$$\begin{aligned}
S_{\text{int}}^{HF}[\phi^*, \phi] &= 2g \int dt \int d^3x n'(\vec{x}, t) (\phi')^*(\vec{x}, t) \phi_0(\vec{x}, t) \\
&+ 2g \int dt \int d^3x n'(\vec{x}, t) \phi'(\vec{x}, t) \phi_0^*(\vec{x}, t) \\
&+ 2g \int dt \int d^3x n'(\vec{x}, t) (\phi')^*(\vec{x}, t) \phi'(\vec{x}, t) \\
&+ g \int dt \int d^3x [m'(\vec{x}, t) (\phi')^*(\vec{x}, t) \phi_0^*(\vec{x}, t) + \text{c.c.}] \\
&+ \frac{g}{2} \int dt \int d^3x [m'(\vec{x}, t) (\phi')^*(\vec{x}, t) (\phi')^*(\vec{x}, t) + \text{c.c.}] \quad (1.3.1)
\end{aligned}$$

that adds to the action $S_0[\phi^*, \phi]$. Here we introduced the complex anomalous density

$$m'(\vec{x}, t) = iG_{12}(\vec{x}, t; \vec{x}, t+) = \langle (\phi')(\vec{x}, t+) (\phi')(\vec{x}, t) \rangle. \quad (1.3.2)$$

In Fig. 1.3 we display how this can be understood diagrammatically.

Performing the same analysis as above, we conclude that the Gross-Pitaevskii equation is modified to

$$\begin{aligned}
i \frac{\partial}{\partial t} \phi_0(\vec{x}, t) &= \left[-\frac{1}{2m} \Delta + V_{\text{ext}}(\vec{x}, t) + 2gn'(\vec{x}, t) + g|\phi_0(\vec{x}, t)|^2 - \mu \right] \phi_0(\vec{x}, t) \\
&+ m'(\vec{x}, t) \phi_0^*(\vec{x}, t), \quad (1.3.3)
\end{aligned}$$

and the normal self-energies are changed into $\Sigma_{11} = \Sigma_{22} = 2g|\phi_0|^2 + 2gn'$ and the anomalous into $\Sigma_{12} = (\Sigma_{21})^* = g(\phi_0)^2 + gm'$. Hence we get for the Bogoliubov-de Gennes equation

$$\begin{pmatrix} h_0 + 2g(n' + |\phi_0|^2) & g(\phi_0)^2 + gm' \\ g(\phi_0^*)^2 + gm'^* & h_0 + 2g(n' + |\phi_0|^2) \end{pmatrix} \cdot \begin{pmatrix} u_n(\vec{x}) \\ v_n(\vec{x}) \end{pmatrix} = \begin{pmatrix} \omega_n & 0 \\ 0 & -\omega_n \end{pmatrix} \cdot \begin{pmatrix} u_n(\vec{x}) \\ v_n(\vec{x}) \end{pmatrix}. \quad (1.3.4)$$

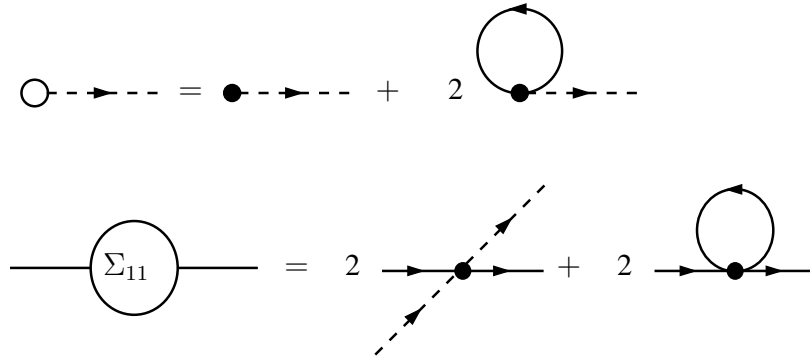


Figure 1.3: The dashed lines correspond to the condensate particles and the solid lines to the non-condensate particles.

These two last equation are known as the Bogoliubov-Hartree-Fock coupled equations.

A simplified version of the Bogoliubov-Hartree-Fock approximation is the Hartree-Fock-Popov approximation [26], where the anomalous density m' is ignored. It is intensely studied in the context of Bose-Einstein condensation in atomic gases, and has been applied to the equilibrium density profile of the gas below the critical temperature [10]. The Popov theory describes the condensate in the presence of a static noncondensed cloud and takes the dynamics of the noncondensed cloud out of consideration. However we will show lateron how to describe the non-equilibrium features of the noncondensed cloud.

Chapter 2

Non-Equilibrium Extension for a System of Bosons in Presence of a Condensate

In Chapter 1 we considered a bosonic gas in a trap. We derived within the Bogoliubov-Hartree-Fock approximation the equation of motion for the macroscopic wave function and the associated Dyson-Beliaev equation for the non-condensate particle Green's function. In this Chapter we want to extend these results for the non-equilibrium case. Non-equilibrium features show up, e.g. when we perturb the system in a non-adiabatic way.

We first present the general idea, how to extend the usual equilibrium field theory to the non-equilibrium case [32, 35, 34], and then proceed similarly in order to find a non-equilibrium description of the results obtained in Chapter 1 (see also [43, 42, 40, 39, 37, 36]).

2.1 Non-Equilibrium Field Theory

2.1.1 The closed time path

The standard construction of the equilibrium many-body theory (see. e.g. [18]) involves the adiabatic switching "on" of interactions at a distant past, and "off" at a distant future. The crucial assumption in this case is that starting from the ground (or equilibrium) state of the system at $t = -\infty$ one reaches the same state at $t = +\infty$ (up to some phase that was acquired along the way).

This is clearly not the case for systems out of equilibrium. If one starts from some arbitrary distribution of states and then switches on and off the interactions, one is going to find the system in a state, which depends, in general, on the characteristics of the switching procedure. The absence of knowledge of the final state ruins the completely the whole construction, since we want to compute averages (or traces) of

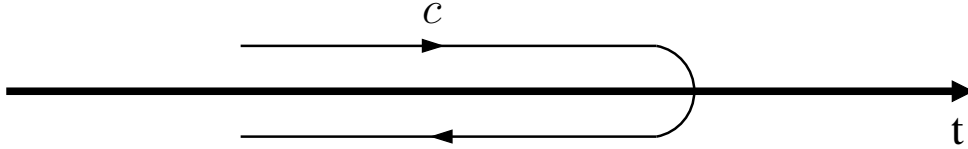


Figure 2.1: The closed time path c

operators corresponding to the physical observables. Thus, one needs to construct a theory which avoids the reference to the final state at $t = \infty$. Nevertheless, we still need to know a final state, since we want to compute averages (or traces). Julian Schwinger suggested to take the final state exactly as the initial one (see also [32]). The main idea is to let the quantum system evolve first in the forward time direction and then to *rewind* its evolution back. At the end one has to construct a theory with the time evolution along the two-branch contour depicted in Fig. 2.1.

This means that no matter what was the final state at $t = +\infty$, after the backward evolution the system will return back to the known initial state at $t = -\infty$. In this construction there is no switching off of the interaction in the far future. Instead of it, both switching on and off take place in the past; on - on the forward branch of the contour and off - on the backward branch. How to construct such a theory and how to use it is the subject of this Section.

In order to construct a theory along the closed time contour we consider a quantum system in thermal equilibrium, which is represented by the time independent Hamiltonian H . Starting from this Hamiltonian the generic non-equilibrium problem can be formulated as follows: far in the past, prior to time t_0 , the system can be thought of as having been brought to the equilibrium state. Then at times larger than $t = t_0$, a time-dependent mechanical perturbation, described by $H'(t)$, is applied to the system. The total Hamiltonian is thus

$$\mathcal{H}(t) = H + H'(t), \quad (2.1.1)$$

where H is the Hamiltonian for the isolated system of interest and $H'(t)$ is a time dependent perturbation acting on it.

The simplest non-equilibrium problem is concerned with the calculation of some average value of a physical quantity A at times $t > t_0$. The unitary transformations relating operators in the Heisenberg pictures governed by the Hamiltonians in $\mathcal{H}(t)$ and H , respectively, is specified by the unitary transformations

$$\mathcal{O}_{\mathcal{H}}(t) = V^\dagger(t, t_0) \mathcal{O}_H(t) V(t, t_0) \quad , \quad V(t, t_0) = T e^{-i \int_{t_0}^t dt' H_{\mathcal{H}}(t')} \quad (2.1.2)$$

and

$$\mathcal{O}_H(t) = U^\dagger(t, t_0) \mathcal{O}_S U(t, t_0) \quad , \quad U(t, t_0) = e^{-iH(t-t_0)}. \quad (2.1.3)$$

where we have chosen t_0 as the reference time where the two pictures coincide. Comparing the two pictures one obtains

$$O_{\mathcal{H}}(t) = U_{\mathcal{H}}^\dagger(t, t_0)U(t_0, t)O_H(t)U^\dagger(t, t_0)U_{\mathcal{H}}(t, t_0), \quad (2.1.4)$$

where

$$U_{\mathcal{H}}(t, t_0) = T e^{-i \int_{t_0}^t dt' \mathcal{H}(t')} \quad (2.1.5)$$

is the time evolution operator corresponding to the Hamiltonian $\mathcal{H}(t)$. Further one observes that the evolution operator $V(t, t_0)$ satisfies the same differential equation and initial condition as $U^\dagger(t, t_0)U_{\mathcal{H}}(t, t_0)$. Thus we obtain the formula

$$V(t, t_0) = U^\dagger(t, t_0)U_{\mathcal{H}}(t, t_0) \quad (2.1.6)$$

or explicitly

$$T e^{-i \int_{t_0}^t dt' H'_H(t')} = e^{iH(t-t_0)} T e^{-i \int_{t_0}^t dt' \mathcal{H}(t')}. \quad (2.1.7)$$

We now consider the contour, the closed time path, introduced above, which starts at t_0 and proceeds along the real time axis to time t and back again to t_0 (Fig. 2.1).

We then show that the transformation between the two Heisenberg pictures, Eq. (2.1.4), can be expressed on closed contour form as

$$O_{\mathcal{H}}(t, t_0) = T_c \left[e^{-i \int_c d\tau H'_H(\tau)} O_H(t) \right] \quad (2.1.8)$$

where τ denotes the contour variable proceeding from t_0 along the real time axis to t . T_c is the contour time ordering operator, that arranges products of operators according to the position of their contour time argument on the closed contour, later contour time places an operator to the left.

For the proof of Eq. (2.1.8) we expand exponential as follows

$$O_{\mathcal{H}}(t, t_0) = \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \int_c d\tau_1 \cdots \int_c d\tau_k T_c [H'_H(\tau_1) \cdots H'_H(\tau_k) O_H(t)] \quad (2.1.9)$$

Now let us only consider the k -th order term. We then split the contour into forward and backward parts

$$c = \vec{c} + \overleftarrow{c}. \quad (2.1.10)$$

The splitting of the contour into forward and backward contours gives 2^k .

$$\begin{aligned} & \int_c d\tau_1 \cdots \int_c d\tau_k T_c [H'_H(\tau_1) \cdots H'_H(\tau_k) O_H(t)] = T_c \left[\left(\int_c d\tau H'_H(\tau) \right)^k O_H(t) \right] \\ &= T_c \left[\left(\int_{\vec{c}} d\tau H'_H(\tau) + \int_{\overleftarrow{c}} d\tau H'_H(\tau) \right)^k O_H(t) \right] \\ &= \sum_{m=0}^k \binom{k}{m} T_{\vec{c}} \left[\left(\int_{\vec{c}} d\tau H'_H(\tau) \right)^k \right] O_H(t) T_{\overleftarrow{c}} \left[\left(\int_{\overleftarrow{c}} d\tau H'_H(\tau) \right)^{k-m} \right] \end{aligned} \quad (2.1.11)$$

Now the summation over k becomes trivial, giving

$$\begin{aligned}
& T_c \left[e^{-i \int_c d\tau H'_H(\tau)} O_H(t) \right] \\
&= \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \sum_{m=0}^k \binom{k}{m} T_{\vec{c}} \left[\left(\int_{\vec{c}} d\tau H'_H(\tau) \right)^k \right] O_H(t) T_{\overleftarrow{c}} \left[\left(\int_{\overleftarrow{c}} d\tau H'_H(\tau) \right)^{k-m} \right] \\
&= T_{\vec{c}} \left[\sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \left(\int_{\vec{c}} d\tau H'_H(\tau) \right)^k \right] O_H(t) T_{\overleftarrow{c}} \left[\sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \left(\int_{\overleftarrow{c}} d\tau H'_H(\tau) \right)^m \right] \\
&= T_{\vec{c}} \left[e^{-i \int_{\vec{c}} d\tau H'_H(\tau)} \right] O_H(t) T_{\overleftarrow{c}} \left[e^{-i \int_{\overleftarrow{c}} d\tau H'_H(\tau)} \right]
\end{aligned} \tag{2.1.12}$$

Parameterizing the forward and backward contours according to

$$\tau(t') = t' \quad t' \in [t_0, t], \tag{2.1.13}$$

we get

$$T_{\vec{c}} \left[e^{-i \int_{\vec{c}} d\tau H'_H(\tau)} \right] = T \left[e^{-i \int_{t_0}^t dt' H'_H(t')} \right] = V(t, t_0) \tag{2.1.14}$$

and

$$T_{\overleftarrow{c}} \left[e^{-i \int_{\overleftarrow{c}} d\tau H'_H(\tau)} \right] = \tilde{T} \left[e^{i \int_{t_0}^t dt' H'_H(t')} \right] = V^\dagger(t, t_0), \tag{2.1.15}$$

i.e. contour ordering along the forward contour is identical to ordinary time ordering, $T_{\vec{c}} = T$, whereas contour ordering along the backward contour corresponds to anti-time ordering, $T_{\overleftarrow{c}} = \tilde{T}$. The equivalence between Eq. (2.1.4) and Eq. (2.1.8) has thus been established.

2.1.2 The closed time path Green's function

Green's functions play a fundamental role in statistical physics since they allow to calculate physical observables.

Now we start by introducing the *lesser* and *greater* Green's functions:

$$G^<(1, 1') = \mp i \left\langle \Psi_{\mathcal{H}}^\dagger(1') \Psi_{\mathcal{H}}(1) \right\rangle \tag{2.1.16}$$

and

$$G^>(1, 1') = -i \left\langle \Psi_{\mathcal{H}}(1) \Psi_{\mathcal{H}}^\dagger(1') \right\rangle, \tag{2.1.17}$$

where $\Psi_{\mathcal{H}}(1)$ is the field operator in the Heisenberg picture with respect to the Hamiltonian $\mathcal{H}(t)$ and where the short-hand notation

$$1 \equiv (t_1, \vec{x}_1) \tag{2.1.18}$$

has been introduced. Here \vec{x}_1 denotes the spatial variable.

Further we define the Green's function on the closed time contour, depicted in Fig (2.1)

$$G(1, 1') = -i \left\langle T_c \left[\Psi_{\mathcal{H}}(1) \Psi_{\mathcal{H}}^\dagger(1') \right] \right\rangle \quad (2.1.19)$$

where T_c is the time ordering operator on the contour c that orders the operators according to the position on the contour of their time arguments

$$T_c \left[\Psi_{\mathcal{H}}(1) \Psi_{\mathcal{H}}^\dagger(1') \right] = \begin{cases} \Psi_{\mathcal{H}}(1) \Psi_{\mathcal{H}}^\dagger(1') & t_1 >_c t_{1'} \\ \pm \Psi_{\mathcal{H}}^\dagger(1') \Psi_{\mathcal{H}}(1) & t_1 <_c t_{1'} \end{cases}, \quad (2.1.20)$$

where $t_1 <_c t_{1'}$ means that t_1 is further along the contour than $t_{1'}$. Comparing the definition of the contour ordering operator with Eq. (2.1.16) and Eq. (2.1.17) we observe that

$$G(1, 1') = \begin{cases} G^>(1, 1') & t_1 >_c t_{1'} \\ G^<(1, 1') & t_1 <_c t_{1'}. \end{cases} \quad (2.1.21)$$

From the equation of motion for the field operator, it can be shown that the equation of motion for the contour-ordered Green's function is

$$\left(i \frac{\partial}{\partial t_1} - h_0(t_1) \right) G(1, 1') = \delta(\vec{x}_1 - \vec{x}_{1'}) \delta_c(t_1 - t_{1'}) - i \left\langle T_c \left([\Psi(1), H_i(1)] \Psi^\dagger(1') \right) \right\rangle \quad (2.1.22)$$

where h_0 denotes the single-particle Hamiltonian, H_i is the many-particle interaction Hamiltonian, and we have introduced the contour delta function

$$\delta_c(t_1 - t_{1'}) = \begin{cases} \delta(t_1 - t_{1'}) & \text{for } t_1 \text{ and } t_{1'} \text{ on upper branch} \\ -\delta(t_1 - t_{1'}) & \text{for } t_1 \text{ and } t_{1'} \text{ on lower branch} \\ 0 & \text{for } t_1 \text{ and } t_{1'} \text{ on different branches} \end{cases}. \quad (2.1.23)$$

The Eq. (2.1.22) will be later the starting point for the derivation of the kinetic equation.

In the proceeding section we will present the perturbation theory that will be used throughout this thesis and show how it is related with the Quantum Boltzmann Equation.

2.1.3 Keldysh formulation and the kinetic equations

In this section we demonstrate how the equation of motion for the closed time path Green's function is reduced to a kinetic equation. For this purpose with start considering the contour-ordered Green's function

$$G(1, 1') = -i \left\langle T_c \left[\Psi_{\mathcal{H}}(1) \Psi_{\mathcal{H}}^\dagger(1') \right] \right\rangle \quad (2.1.24)$$

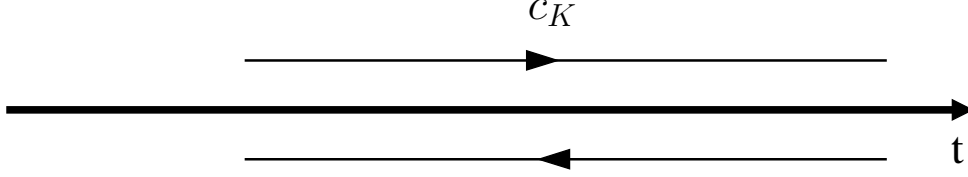


Figure 2.2: The Keldysh contour c_K

Then we make use of Eq. (2.1.8) that relates the two Heisenberg pictures on the closed time contour and rewrite G as follows

$$\begin{aligned}
G(1, 1') &= -i \left\langle T_{c_1} \left[e^{-i \int_{c_1} d\tau H'_H(\tau)} \Psi_H(1) \right] T_{c_{1'}} \left[e^{-i \int_{c_{1'}} d\tau H'_H(\tau)} \Psi_H^\dagger(1') \right] \right\rangle \\
&= -i \left\langle T_{c_1+c_{1'}} \left[e^{-i \int_{c_1+c_{1'}} d\tau H'_H(\tau)} \Psi_H(1) \Psi_H^\dagger(1') \right] \right\rangle \\
&= -i \left\langle T_c \left[e^{-i \int_c d\tau H'_H(\tau)} \Psi_H(1) \Psi_H^\dagger(1') \right] \right\rangle, \tag{2.1.25}
\end{aligned}$$

where in the first line the contour c_1 ($c_{1'}$) starts at t_0 and passes through t_1 ($t_{1'}$), respectively, and returns to t_0 . In the last equality we introduced the combined contour c , which starts at t_0 and stretches through $\max(t_1, t_{1'})$ (or all the way to $+\infty$) and back again to t_0 .

Comparing Eq. (2.1.25) and its equilibrium field theory analog we observe that the nonequilibrium version of the Green's function the expectation value

$$\left\langle T_c \left[\exp \left(-i \int_c d\tau H'_H(\tau) \right) \right] \right\rangle$$

is missing in the denominator. The absence of the denominator suggests that the non-equilibrium perturbation theory has a simpler structure than the standard equilibrium theory as there is no need for canceling of unlinked or disconnected diagrams.

Since we are not interested in the initial correlations we can let t_0 to minus infinity, $t_0 \rightarrow -\infty$, and we then obtain the contour c_K introduced by Keldysh (1964). The contour c_K depicted in Fig. (2.2) consists of two parts: c_1 extending from $-\infty$ to $+\infty$ and c_2 extending from $+\infty$ to $-\infty$.

The contour-ordered Green's function G_{c_K} specified by the Keldysh contour can then be mapped onto the Keldysh space,

$$G_{c_K}(1, 1') \mapsto \widehat{G} \equiv \begin{pmatrix} \widehat{G}_{11} & \widehat{G}_{12} \\ \widehat{G}_{21} & \widehat{G}_{22} \end{pmatrix}, \tag{2.1.26}$$

by the prescription that the ij component of \widehat{G} can be defined as $G_{c_K}(1, 1')$ for t_1 and $t_{1'}$ residing on c_i and c_j , respectively. \widehat{G}_{11} is the usual time-ordered Green's

function and \widehat{G}_{22} denotes the anti-time-ordered Green's function. The remaining two components \widehat{G}_{12} and \widehat{G}_{21} are the lesser and greater Green's functions introduced in the Section 2.1.2, respectively:

$$\begin{aligned}\widehat{G}_{11}(1, 1') &= -i \left\langle T \left[\Psi_{\mathcal{H}}(1) \Psi_{\mathcal{H}}^\dagger(1') \right] \right\rangle, \\ \widehat{G}_{12}(1, 1') &= G^<(1, 1'), \\ \widehat{G}_{21}(1, 1') &= G^>(1, 1'), \\ \widehat{G}_{22}(1, 1') &= -i \left\langle \widetilde{T} \left[\Psi_{\mathcal{H}}(1) \Psi_{\mathcal{H}}^\dagger(1') \right] \right\rangle.\end{aligned}\tag{2.1.27}$$

Further, the components of \widehat{G} are not linearly independent and from their definition one obtains the relation

$$\widehat{G}_{11} + \widehat{G}_{22} = \widehat{G}_{12} + \widehat{G}_{21}.\tag{2.1.28}$$

Moreover, performing a rotation in Keldysh space it is possible to get rid off the redundant component:

$$\widehat{G} \longmapsto \begin{pmatrix} 0 & G^{\text{av}} \\ G^{\text{ret}} & G^K \end{pmatrix},\tag{2.1.29}$$

where, besides the usual retarded and advanced Green's functions

$$\begin{aligned}G^{\text{ret}}(1, 1') &= \theta(t_1 - t_{1'}) [G^>(1, 1') - G^<(1, 1')], \\ G^{\text{av}}(1, 1') &= -\theta(t_{1'} - t_1) [G^>(1, 1') - G^<(1, 1')],\end{aligned}\tag{2.1.30}$$

there is the Keldysh component defined by:

$$G^K(1, 1') = G^>(1, 1') + G^<(1, 1')\tag{2.1.31}$$

which is central to the non-equilibrium formulation.

In condensed matter physics a representation in terms of trigonal matrices is often used. To obtain this representation we first perform the transformation in Keldysh space

$$\check{G} = \tau_3 \widehat{G}\tag{2.1.32}$$

followed by a rotation

$$G = R \check{G} R^\dagger\tag{2.1.33}$$

where

$$R = \frac{1}{\sqrt{2}}(1 - i\tau_2),\tag{2.1.34}$$

and τ_i ($i = 1, 2, 3$) are the Pauli matrices. Hence, we obtain for G :

$$G = \begin{pmatrix} G^{\text{ret}} & G^K \\ 0 & G^{\text{av}} \end{pmatrix}.\tag{2.1.35}$$

The advantage of these economical representations are the functions G^{ret} and G^K appearing in its components, which contain distinctly physical relevant information: the spectral function, related to G^{ret} has the information about the quantum states of a system, the energy spectrum, and the kinetic Green's function, G^K , has the information about the occupation of these states for non-equilibrium situations.

Now, considering a system of interacting particles, perturbation theory in H_{int} can be performed analogously to the equilibrium theory. The diagrammatic expansion in the Keldysh formulation is similar to the standard Feynman representation. The difference arises from the contour integration, which corresponds to a summation over the upper and lower branches at each internal vertex:

$$\int_c d\tau = \int_{-\infty}^{\infty} dt - \int_{-\infty}^{\infty} dt. \quad (2.1.36)$$

Thus the *right-hand* Dyson equation becomes a matrix equation in Keldysh space

$$\int d^3x_2 \int_{-\infty}^{\infty} dt_2 [G_0^{-1}(1,2) - \Sigma(1,2)] G(2,1') = \delta(x_1 - x_{1'})\delta(t_1 - t_{1'}), \quad (2.1.37)$$

and even more compact

$$\int d2 [G_0^{-1}(1,2) - \Sigma(1,2)] G(2,1') = \delta(1 - 1'). \quad (2.1.38)$$

Here G_0^{-1} is the inverse matrix of the unperturbed Green's function:

$$G_0^{-1}(1,1') = \left[i \frac{\partial}{\partial t_1} - h_0(1) \right] \delta(1 - 1'), \quad (2.1.39)$$

and

$$\Sigma = \begin{pmatrix} \Sigma^{\text{ret}} & \Sigma^K \\ 0 & \Sigma^{\text{av}} \end{pmatrix}. \quad (2.1.40)$$

consists of the usual retarded and advanced selfenergies

$$\begin{aligned} \Sigma^{\text{ret}}(1,1') &= \theta(t_1 - t_{1'}) [\Sigma^>(1,1') - \Sigma^<(1,1')] \\ \Sigma^{\text{av}}(1,1') &= -\theta(t_{1'} - t_1) [\Sigma^>(1,1') - \Sigma^<(1,1')], \end{aligned} \quad (2.1.41)$$

and a Keldysh component

$$\Sigma^K(1,1') = \Sigma^>(1,1') + \Sigma^<(1,1'). \quad (2.1.42)$$

In the same way one can get the *left-hand* or conjugate Dyson equation

$$\int d2 G(1',2) [(G_0^{-1})^*(2,1) - \Sigma(2,1)] = \delta(1' - 1). \quad (2.1.43)$$

Moreover, we observe that the *lesser* Green's function $G^<$ is the most closely related to the density matrix. We will make use of this fact in order to derive the *Quantum Boltzmann equation*. Therefore we start considering the lesser component of the right-hand Dyson equation (Eq. (2.1.38)) and its conjugate (Eq. (2.1.43)) and subtract them from each other:

$$[(G_0^{-1})^*(1') - G_0^{-1}(1)] G^<(1, 1') = I[g, \Sigma] \quad (2.1.44)$$

with

$$\begin{aligned} I[g, \Sigma] = & \int d2 [G^{\text{ret}}(1, 2)\Sigma^<(2, 1') + G^<(1, 2)\Sigma^{\text{av}}(2, 1') \\ & - \Sigma^{\text{ret}}(1, 2)G^<(2, 1') - \Sigma^<(1, 2)G^{\text{av}}(2, 1')] \end{aligned} \quad (2.1.45)$$

as the inverse fourier transform of the collision Integral $C(n)$ on the right-hand side of the Boltzmann equation, as we will see. We also used, that

$$\begin{aligned} G_0^{-1}(1, 1') &= G_0^{-1}(1)\delta(1 - 1') \\ &= \left[i\frac{\partial}{\partial t_1} - h_0(1) \right] \delta(1 - 1') \end{aligned} \quad (2.1.46)$$

in order to get rid off the integral on the left-hand side of the equation.

The operator acting on $G^<$ on the left-hand side of Eq. (2.1.44) is

$$\begin{aligned} (G_0^{-1})^*(1') - G_0^{-1}(1) &= -i \left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_{1'}} \right) - \frac{1}{2m}(\Delta_1 - \Delta_{1'}) \\ &= -i \left(\frac{\partial}{\partial T} - \frac{i}{m} \nabla_R \cdot \nabla_r \right). \end{aligned} \quad (2.1.47)$$

In the last line the Wigner transformation was performed, introducing the coordinates

$$\vec{R} = \frac{\vec{x}_1 + \vec{x}_{1'}}{2}, \quad \vec{r} = \vec{x}_1 - \vec{x}_{1'}, \quad (2.1.48)$$

and

$$T = \frac{t_1 + t_{1'}}{2}, \quad t = t_1 - t_{1'}. \quad (2.1.49)$$

where T and \vec{R} describe the macroscopic properties, governed by the non-equilibrium features under consideration, and t and \vec{r} describe the microscopic properties, governed by the characteristics of the system.

We have mentioned above that the lesser Green's function is the most closely related to the density matrix. Let us now specify this relation. For this purpose we write the $G^<$ in terms of (T, \vec{R}) and (t, \vec{r}) :

$$\begin{aligned} G^<(1, 1') &= G^<(T + \frac{t}{2}, \vec{R} + \frac{\vec{r}}{2}; T - \frac{t}{2}, \vec{R} - \frac{\vec{r}}{2}) \\ &\equiv G^<(T, \vec{R}; t, \vec{r}), \end{aligned} \quad (2.1.50)$$

and perform Fourier transformation with respect to t and \vec{r} :

$$G^<(T, \vec{R}; \omega, \vec{p}) = \int d^3r dt e^{-i(\vec{x}\cdot\vec{p}-\omega t)} G^<(T, \vec{R}; t, \vec{r}). \quad (2.1.51)$$

Then we get for the distribution function

$$n(T, \vec{R}, \vec{p}) = -i \int \frac{d\omega}{2\pi} G^<(T, \vec{R}; \omega, \vec{p}); \quad (2.1.52)$$

the integration over $d\omega/2\pi$ is equivalent to setting $t = 0$.

Now returning our attention to Eq. (2.1.44) we take the Fourier components (2.1.51) on each side and put $t = 0$. Making use of the definition of the distribution function (see Eq. (2.1.52)) we obtain

$$\left[\frac{\partial}{\partial T} + \frac{\vec{p}}{m} \nabla_r \right] n(T, \vec{R}, \vec{p}) = C[n], \quad (2.1.53)$$

where $C[n]$ is the collision integral and is some functional of the distribution function. And from this equation we obtain the relation between $C[n]$ and $I[g, \Sigma]$.

In the coming Section we will apply the formalism derived here to a system of weakly interacting bosons at zero temperature. This would be the non-equilibrium extension of the theory presented in Chapter 1.

2.2 Weakly Interacting Bose Gas Out of Equilibrium

In section 1.3 we presented the derivation of the Dyson equation for a system of trapped interacting bosons at zero temperature in Bogoliubov-Hartree-Fock approximation. Due to the finite value of the order parameter, we found that the off-diagonal elements of the Green's function for the non-condensate particles do not vanish. As a consequence, the Green's function becomes a 2×2 matrix in the so called Bogoliubov space, where the off-diagonal elements are the anomalous Green's functions. Since we want to study non-equilibrium features of two coupled Bose-Einstein condensates, we have to extend the theory presented in section 1.3 to the non-equilibrium case. What we basically do is to combine the Keldysh technique for systems out of equilibrium with the field theory for bosonic systems at zero temperature.

Now we perform the derivation of the kinetic equations and its associated equations for the macroscopic wave functions. For this purpose we make use of the approach suggested by J. W. Kane and L. P. Kadanoff in 1965 (see [36]), which was developed with the specific goal of deriving the phenomenological two-fluid hydrodynamical equations of Landau.

Before we start with the derivation of the equations of motion for the two macroscopic condensate wave functions and the associated *Quantum-Boltzmann* equations for the non-condensate particles, we introduce the 2×2 matrix single-particle Green's function

$$\mathbf{G}(1, 1') = -i \begin{pmatrix} \langle T_c \Psi(1) \Psi^\dagger(1') \rangle & \langle T_c \Psi(1) \Psi(1') \rangle \\ \langle T_c \Psi^\dagger(1) \Psi^\dagger(1') \rangle & \langle T_c \Psi^\dagger(1) \Psi(1') \rangle \end{pmatrix}. \quad (2.2.1)$$

where we used the short hand notation $1 \equiv (\vec{x}, t)$ and $1' \equiv (\vec{x}', t')$. Here, as before, T_c denotes the time ordering along the Keldysh contour. The real time response functions $\mathbf{G}^<$ and $\mathbf{G}^>$ are given by

$$\mathbf{G}^<(1, 1') = -i \begin{pmatrix} \langle \Psi^\dagger(1') \Psi(1) \rangle & \langle \Psi(1') \Psi(1) \rangle \\ \langle \Psi^\dagger(1') \Psi^\dagger(1) \rangle & \langle \Psi(1') \Psi^\dagger(1) \rangle \end{pmatrix} \quad (2.2.2)$$

and

$$\mathbf{G}^>(1, 1') = -i \begin{pmatrix} \langle \Psi(1) \Psi^\dagger(1') \rangle & \langle \Psi(1) \Psi(1') \rangle \\ \langle \Psi^\dagger(1) \Psi^\dagger(1') \rangle & \langle \Psi^\dagger(1) \Psi(1') \rangle \end{pmatrix} \quad (2.2.3)$$

For a system consisting of bosons at zero temperature we have pointed out in Chapter 1 that is more convenient to separate out the part of the field operator corresponding to the condensed phase

$$\Psi(1) = \Psi_0(1) + \Psi'(1), \quad (2.2.4)$$

where $\Psi_0(1) = \langle \Psi(1) \rangle$ is the macroscopic condensate wave function and Ψ' is the annihilation operator of the non-condensate particles. Now, using the decomposition (2.2.4), the matrix propagator (2.2.1) splits into two parts

$$\mathbf{G}(1, 1') = \mathbf{G}'(1, 1') + \mathbf{C}(1, 1'). \quad (2.2.5)$$

Here is \mathbf{G}' identical to (2.2.1), except that it involves the non-condensate part of the field operators, and the condensate part is given by

$$\mathbf{C}(1, 1') = -i \begin{pmatrix} \Psi_0(1) \Psi_0^*(1') & \Psi_0(1) \Psi_0(1') \\ \Psi_0^*(1) \Psi_0^*(1') & \Psi_0^*(1) \Psi_0(1') \end{pmatrix}. \quad (2.2.6)$$

In order to derive the system of self-consistent equations for the Green's functions, we consider the standard Heisenberg equations of motion for the field operators Ψ and Ψ^\dagger

$$\left(i \frac{\partial}{\partial t_1} - h(1) \right) \Psi(1) = g \Psi^\dagger(1) \Psi(1) \Psi(1) \quad (2.2.7)$$

and

$$\left(-i \frac{\partial}{\partial t_1} - h(1) \right) \Psi^\dagger(1) = g \Psi^\dagger(x, t) \Psi^\dagger(1) \Psi(1), \quad (2.2.8)$$

where

$$h(1) = -\frac{1}{2m}\Delta + V_{\text{ext}} - \mu \quad (2.2.9)$$

is the one particle Hamiltonian.

For the derivation of the equation for the macroscopic wave function we apply the time ordering operator on the Keldysh contour T_c to both sides of Eq. (2.2.7) and then take its average

$$\left(i\frac{\partial}{\partial t_1} - h(1)\right) \langle T_c \Psi(1) \rangle = g \langle T_c \Psi(1) \Psi(1) \Psi^\dagger(1+) \rangle. \quad (2.2.10)$$

We now can identify $\langle T_c \Psi(1) \rangle$ with the order parameter $\Psi_0(1)$. On the right hand side of the equation we got the time ordered average on the Keldysh contour three bosons field operators. We make use of the decomposition (2.2.4) and write for this average

$$\begin{aligned} \langle T_c \Psi(1) \Psi(1) \Psi^\dagger(1+) \rangle &= |\Psi_0(1)|^2 \Psi_0(1) + 2 \langle T_c \Psi'(1) \Psi'^\dagger(1+) \rangle \Psi_0(1) + \\ &+ \langle \Psi'(1) \Psi'(1) \rangle \Psi_0^*(1) + \langle \Psi'(1) \Psi'(1) \Psi'^\dagger(1+) \rangle. \end{aligned} \quad (2.2.11)$$

Here we used the fact, that the average over one field operator corresponding to the non-condensate particles vanish. We include the average containing three operators Ψ' , since it appears, as we have pointed out before, as a second order process and is responsible for thermalization effects.

Inserting Eq. (2.2.11) in the equation for the order parameter (2.2.10), we obtain the equation for the Green's function \mathbf{C} of the condensate particles

$$\int_c d2 [\mathbf{G}_0^{-1}(1, 2) - \mathbf{S}^{HF}(1, 2)] \mathbf{C}(2, 1') = \int_c d2 \mathbf{S}(1, 2) \mathbf{C}(2, 1') \quad (2.2.12)$$

where the time integration is performed along the Keldysh contour. Here the bare 2×2 matrix propagator \mathbf{G}_0 for bosons is defined by

$$\mathbf{G}_0^{-1}(1, 1') = \left[i\tau_3 \frac{\partial}{\partial t_1} - h(1) + \mu \right] \mathbf{1} \delta(1 - 1'), \quad (2.2.13)$$

where

$$\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (2.2.14)$$

and $\mathbf{1}$ is the 2×2 identity matrix. The first order perturbation theory is described by the self-energy matrix \mathbf{S}^{HF} in Hartree-Fock approximation

$$\mathbf{S}^{HF}(1, 1') = ig \begin{pmatrix} 2G'_{11}{}^<(1, 1+) + C_{11}(1, 1) & G'_{12}{}^<(1, 1) + C_{12}(1, 1) \\ G'_{21}{}^<(1, 1) + C_{21}(1, 1) & 2G'_{11}{}^<(1, 1+) + C_{11}(1, 1) \end{pmatrix} \delta(1 - 1') \quad (2.2.15)$$

where $G'_{ij}{}^{<}$ and C_{ij} are the components of $\mathbf{G}'^{<}$ and \mathbf{C} , respectively. The second and higher order contributions are involved in the selfenergy \mathbf{S} .

Now we make use of the analytical continuation (see Appendix A) procedure in order to obtain the real time version of Eq. (2.2.12)

$$\int d2 [\mathbf{G}_0^{-1}(1, 2) - \mathbf{S}^{HF}(1, 2)] \mathbf{C}(2, 1') = \int_{-\infty}^{t_1} d2 [\mathbf{S}^>(1, 2) - \mathbf{S}^<(1, 2)] \mathbf{C}(2, 1'), \quad (2.2.16)$$

where the time integration without any boundaries extends now from $t = -\infty$ to $t = \infty$. Having now the equation of motion for the condensate Green's function, we can immediately write down the *generalized Gross-Pitaevskii* equation

$$\begin{aligned} & \left[i \frac{\partial}{\partial t_1} + \frac{1}{2m} \Delta - V_{\text{ext}} + \mu - g(n_0(1) - 2n'(1)) \right] \Psi_0(1) = gm'(1) \Psi_0^*(1) \\ & + \int_{-\infty}^{t_1} d2 [S_{11}^>(1, 2) - S_{11}^<(1, 2)] \Psi_0(2) + \int_{-\infty}^{t_1} d2 [S_{12}^>(1, 2) - S_{12}^<(1, 2)] \Psi_0^*(2), \end{aligned} \quad (2.2.17)$$

where $n_0(1) = |\Psi_0(1)|^2$ and $n'(1) = iG'_{11}{}^{<}(1, 1+)$ are the density of the condensate phase and the density of the non-condensate particles, respectively. Moreover,

$$m'(1) = iG'_{12}{}^{<}(1, 1) \quad (2.2.18)$$

is the anomalous density. As we have mentioned before, in order to describe the dynamics of the non-condensate particles, one needs to derive a quantum Boltzmann equation for the distribution function of the excited particles. For this purpose we derive the Dyson equations for the Bose propagator (2.2.1) from the equation of motion for the Green's function. Then we subtract Eq. (2.2.12) from this equation and end up with the Dyson-Belyayev equation for the non-condensate particle propagator

$$\int_c d2 [\mathbf{G}_0^{-1}(1, 2) - \mathbf{\Sigma}^{HF}(1, 2)] \mathbf{G}'(2, 1') = \mathbb{1} \delta(1 - 1') + \int_c d2 \mathbf{\Sigma}_c(1, 2) \mathbf{G}'(2, 1'). \quad (2.2.19)$$

where

$$\mathbf{\Sigma}^{HF}(1, 1') = ig \begin{pmatrix} 2[G'_{11}{}^{<}(1, 1+) + C_{11}(1, 1)] & G'_{12}{}^{<}(1, 1) + C_{12}(1, 1) \\ G'_{21}{}^{<}(1, 1) + C_{21}(1, 1) & 2[G'_{11}{}^{<}(1, 1+) + C_{11}(1, 1)] \end{pmatrix} \delta(1 - 1') \quad (2.2.20)$$

is the lowest order contribution described in Bogoliubov-Hartree-Fock approximation, and $\mathbf{\Sigma}_c$ involves the second order and higher order processes and where the

index c refers to the collisions. We also recall that the single particle selfenergy which is involved in Eq. (2.2.19) has been split into two parts [33]

$$\Sigma(1, 1') = \Sigma_c(1, 1') + \Sigma^{HF}(1, 1'). \quad (2.2.21)$$

We now treat the right hand side of Eq. (2.2.19) like above and make use of the analytical continuation procedure, with the goal to end up with its real time version

$$\begin{aligned} & \int d2 [\mathbf{G}_0^{-1}(1, 2) - \Sigma^{HF}(1, 2)] \mathbf{G}'^{\geq}(2, 1') \\ &= \int_{-\infty}^{t_1} d2 \Gamma(1, 2) \mathbf{G}'^{\geq}(2, 1') - \int_{-\infty}^{t_1'} d2 \Sigma_c^{\geq}(1, 2) \mathbf{A}(2, 1') \end{aligned} \quad (2.2.22)$$

and its hermitian conjugated

$$\begin{aligned} & \int d2 \mathbf{G}'^{\leq}(1, 2) [(\mathbf{G}_0^{-1})^*(2, 1') - \Sigma^{HF}(2, 1')] \\ &= \int_{-\infty}^{t_1} d2 \mathbf{A}(1, 2) \Sigma_c^{\leq}(2, 1') - \int_{-\infty}^{t_1'} d2 \mathbf{G}'^{\leq}(1, 2) \Gamma(2, 1'), \end{aligned} \quad (2.2.23)$$

where we have introduced the spectral weight function

$$\mathbf{A}(1, 1') = \mathbf{G}'^{>}(1, 1') - \mathbf{G}'^{<}(1, 1') \quad (2.2.24)$$

and similarly for the self-energy

$$\Gamma(1, 1') = \Sigma^{>}(1, 1') - \Sigma^{<}(1, 1'). \quad (2.2.25)$$

We have thus obtained the nonequilibrium form of the Dyson-Beliaev equations of motion for the non-condensate particles. These equations are the starting point for the computation of the quantum Boltzmann equations. For the derivation we first write down the "11" and "12" components of Eq. (2.2.22) and Eq. (2.2.23)

$$\begin{aligned} & \left[i \frac{\partial}{\partial t_1} - h(1) \right] G'_{11}^{<}(1, 1') = \Sigma_{11}^{HF}(1) G'_{11}^{<}(1, 1') + \Sigma_{12}^{HF}(1) G'_{21}^{<}(1, 1') \\ & + \int_{-\infty}^{t_1} d2 [\Gamma_{11}(1, 2) G'_{11}^{<}(2, 1') + \Gamma_{12}(1, 2) G'_{21}^{<}(2, 1')] \\ & - \int_{-\infty}^{t_1'} d2 [\Sigma_{11}^{c<}(1, 2) A_{11}(2, 1') + \Sigma_{12}^{c<}(1, 2) A_{21}(2, 1')], \end{aligned} \quad (2.2.26)$$

$$\begin{aligned}
\left[i \frac{\partial}{\partial t_1} - h(1) \right] G'_{12}{}^{\leq}(1, 1') &= \Sigma_{11}^{HF}(1) G'_{12}{}^{\leq}(1, 1') + \Sigma_{12}^{HF}(1) G'_{22}{}^{\leq}(1, 1') \\
&+ \int_{-\infty}^{t_1} d2 [\Gamma_{11}(1, 2) G'_{12}{}^{\leq}(2, 1') + \Gamma_{12}(1, 2) G'_{22}{}^{\leq}(2, 1')] \\
&- \int_{-\infty}^{t_1'} d2 [\Sigma_{11}^{c\leq}(1, 2) A_{12}(2, 1') + \Sigma_{12}^{c\leq}(1, 2) A_{22}(2, 1')], \tag{2.2.27}
\end{aligned}$$

and

$$\begin{aligned}
\left[-i \frac{\partial}{\partial t_{1'}} - h(1') \right] G'_{11}{}^{\leq}(1, 1') &= \Sigma_{11}^{HF}(1') G'_{11}{}^{\leq}(1, 1') + \Sigma_{21}^{HF}(1') G'_{12}{}^{\leq}(1, 1') \\
&- \int_{-\infty}^{t_1'} d2 [G'_{11}{}^{\leq}(1, 2) \Gamma_{11}(2, 1') + G'_{12}{}^{\leq}(1, 2) \Gamma_{21}(2, 1')] \\
&+ \int_{-\infty}^{t_1} d2 [A_{11}(1, 2) \Sigma_{11}^{c\leq}(2, 1') + A_{12}(1, 2) \Sigma_{21}^{c\leq}(2, 1')], \tag{2.2.28}
\end{aligned}$$

$$\begin{aligned}
\left[i \frac{\partial}{\partial t_{1'}} - h(1') \right] G'_{12}{}^{\leq}(1, 1') &= \Sigma_{22}^{HF}(1') G'_{12}{}^{\leq}(1, 1') + \Sigma_{12}^{HF}(1') G'_{11}{}^{\leq}(1, 1') \\
&- \int_{-\infty}^{t_1'} d2 [G'_{12}{}^{\leq}(1, 2) \Gamma_{22}(2, 1') + G'_{11}{}^{\leq}(1, 2) \Gamma_{12}(2, 1')] \\
&+ \int_{-\infty}^{t_1} d2 [A_{11}(1, 2) \Sigma_{12}^{c\leq}(2, 1') + A_{12}(1, 2) \Sigma_{22}^{c\leq}(2, 1')]. \tag{2.2.29}
\end{aligned}$$

The equations for the greater component of the Bose propagator take the same form except that $>$ appears instead of $<$ as superscript.

The equations of motion (2.2.26)-(2.2.29) are formally exact and can be used to derive a generalized Boltzmann equation. For this purpose we use the Wigner transformation introduced in Eq. (2.1.48) and Eq. (2.1.49) in order to separate the variables, (\vec{r}, t) , describing the microscopic properties of the system, from the variables, (\vec{R}, T) , describing the macroscopic properties. For the derivation of the Boltzmann equations, one needs to rewrite the equations for the real time non-equilibrium Green's functions G_{11}^{\leq} and G_{12}^{\leq} in the Wigner representation. To this

end, we first subtract Eq. (2.2.28) from Eq. (2.2.26)

$$\begin{aligned}
& \left[i \left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_{1'}} \right) - (h(1) - h(1')) - (\Sigma_{11}^{HF}(1) - \Sigma_{11}^{HF}(1')) \right] G'_{11}(1, 1') \\
&= \Sigma_{12}^{HF}(1) G'_{21}(1, 1') - \Sigma_{21}^{HF}(1') G'_{12}(1, 1') \\
&+ \int_{-\infty}^{t_1} d2 [\Gamma_{11}(1, 2) G'_{11}(2, 1') - A_{11}(1, 2) \Sigma_{11}^{c<}(2, 1')] \\
&+ \int_{-\infty}^{t_1} d2 [\Gamma_{12}(1, 2) G'_{21}(2, 1') - A_{12}(1, 2) \Sigma_{21}^{c<}(2, 1')] \\
&+ \int_{-\infty}^{t_{1'}} d2 [G'_{12}(1, 2) \Gamma_{21}(2, 1') - \Sigma_{12}^{c<}(1, 2) A_{21}(2, 1')] \\
&+ \int_{-\infty}^{t_{1'}} d2 [G'_{11}(1, 2) \Gamma_{11}(2, 1') - \Sigma_{11}^{c<}(1, 2) A_{11}(2, 1')] \tag{2.2.30}
\end{aligned}$$

and then take the sum of Eq. (2.2.27) and Eq. (2.2.29)

$$\begin{aligned}
& \left[i \left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_{1'}} \right) - (h(1) + h(1')) - (\Sigma_{11}^{HF}(1) + \Sigma_{22}^{HF}(1')) \right] G'_{12}(1, 1') \\
&= \Sigma_{12}^{HF}(1) G'_{22}(1, 1') + \Sigma_{12}^{HF}(1') G'_{11}(1, 1') \\
&+ \int_{-\infty}^{t_1} d2 [\Gamma_{11}(1, 2) G'_{12}(2, 1') + A_{12}(1, 2) \Sigma_{22}^{c<}(2, 1')] \\
&+ \int_{-\infty}^{t_1} d2 [\Gamma_{12}(1, 2) G'_{22}(2, 1') + A_{11}(1, 2) \Sigma_{12}^{c<}(2, 1')] \\
&- \int_{-\infty}^{t_{1'}} d2 [G'_{12}(1, 2) \Gamma_{22}(2, 1') + \Sigma_{11}^{c<}(1, 2) A_{12}(2, 1')] \\
&- \int_{-\infty}^{t_{1'}} d2 [G'_{11}(1, 2) \Gamma_{12}(2, 1') + \Sigma_{12}^{c<}(1, 2) A_{22}(2, 1')] \tag{2.2.31}
\end{aligned}$$

Now we turn our attention to the left hand side of Eq. (2.2.30) and Eq. (2.2.31) and express them in terms of the center of mass coordinates (\vec{R}, T) and the relative

coordinates (\vec{r}, t)

$$\left[i \frac{\partial}{\partial T} + \frac{\nabla_R \cdot \nabla_r}{m} - \left[\left(\vec{r} \cdot \nabla_R + t \frac{\partial}{\partial T} \right) \left(V_{\text{ext}}(\vec{R}, T) + \Sigma_{11}^{HF}(\vec{R}, t) \right) \right] \right] G'_{11}{}^{<}(\vec{r}, t; \vec{R}, T) \quad (2.2.32)$$

and

$$\left[i \frac{\partial}{\partial T} + \frac{\Delta_R}{4m} + \frac{\Delta_r}{m} - 2V_{\text{ext}}(\vec{R}, T) - 2\Sigma_{11}^{HF}(\vec{R}, t) \right] G'_{12}{}^{<}(\vec{r}, t; \vec{R}, T) \quad (2.2.33)$$

where we used the fact that $G'_{ij}{}^{<}(1, 1') \equiv G'_{ij}{}^{<}(\vec{r}, t; \vec{R}, T)$. Further we made the assumption, that under the non-equilibrium perturbation $G'_{ij}{}^{<}(1, 1')$ and $G'_{ij}{}^{>}(1, 1')$ are slowly varying functions of the coordinates (\vec{R}, T) but sharply peaked about the zero values of (\vec{r}, t) . Thus the Hartree-Fock contribution of the selfenergy Σ_{ij}^{HF} and the trapping potential V_{ext} depend only on small values of (\vec{r}, t) , and we can therefore expand them in powers of \vec{r} and t , keeping only the linear term. This is a suitable approximation for the low density limit, which is case we study in this thesis (see also [33]).

From now on we replace the integrals appearing on the right hand side of Eq. (2.2.30) and Eq. (2.2.31) by the functions $I_{11}^{<}$ and $I_{12}^{<}$, respectively. Here the superscript means, that the quantities appearing in the integral are lesser quantities. Each of these functions depends on the coordinates 1 and 1'. For a more detailed discussion of these integrals see Appendix B. Further we should keep in mind that each of these functions can be decomposed into two terms; one describing collisions between non-condensate particles and other describing between condensate and non-condensate particles. Hence, Eq. (2.2.30) and Eq. (2.2.31) become

$$\begin{aligned} & \left[i \frac{\partial}{\partial T} + \frac{\nabla_R \cdot \nabla_r}{m} - \left[\left(\vec{r} \cdot \nabla_R + t \frac{\partial}{\partial T} \right) \left(V_{\text{ext}}(\vec{R}, T) + \Sigma_{11}^{HF}(\vec{R}, T) \right) \right] \right] G'_{11}{}^{<}(\vec{r}, t; \vec{R}, T) \\ &= \Sigma_{12}^{HF}(\vec{R}, T) G'_{21}{}^{<}(\vec{r}, t; \vec{R}, T) - \Sigma_{21}^{HF}(\vec{R}, T) G'_{12}{}^{<}(\vec{r}, t; \vec{R}, T) \\ &+ \frac{1}{2} \left[G'_{21}{}^{<}(\vec{r}, t; \vec{R}, T) \left(\vec{r} \cdot \nabla_R + t \frac{\partial}{\partial T} \right) \Sigma_{12}^{HF}(\vec{R}, T) + \right. \\ &+ G'_{12}{}^{<}(\vec{r}, t; \vec{R}, T) \left. \left(\vec{r} \cdot \nabla_R + t \frac{\partial}{\partial T} \right) \Sigma_{21}^{HF}(\vec{R}, T) \right] \\ &+ I_{11}^{<}(\vec{r}, t; \vec{R}, T), \end{aligned} \quad (2.2.34)$$

$$\begin{aligned}
& \left[i \frac{\partial}{\partial T} + \frac{\Delta_R}{4m} + \frac{\Delta_r}{m} - 2V_{\text{ext}}(\vec{R}, T) - 2\Sigma_{11}^{HF}(\vec{R}, t) \right] G'_{12}{}^<(\vec{r}, t; \vec{R}, T) \\
&= \Sigma_{12}^{HF}(\vec{R}, T) G'_{22}{}^<(\vec{r}, t; \vec{R}, T) + \Sigma_{12}^{HF}(\vec{R}, T) G'_{11}{}^<(\vec{r}, t; \vec{R}, T) \\
&+ \frac{1}{2} \left[G'_{22}{}^<(\vec{r}, t; \vec{R}, T) \left(\vec{r} \cdot \nabla_R + t \frac{\partial}{\partial T} \right) \Sigma_{12}^{HF}(\vec{R}, T) + \right. \\
&- G'_{11}{}^<(\vec{r}, t; \vec{R}, T) \left. \left(\vec{r} \cdot \nabla_R + t \frac{\partial}{\partial T} \right) \Sigma_{12}^{HF}(\vec{R}, T) \right] \\
&+ I_{12}{}^<(\vec{r}, t; \vec{R}, T) \tag{2.2.35}
\end{aligned}$$

respectively. Here we expanded the Hartree-Fock self-energies on the right hand side of the equation in powers of \vec{r} and t , and kept only the linear terms.

In section 2.1.3 we mentioned that the lesser Green's function is the most closely related to the density matrix. We also specified its relation to the distribution function (see Eq. (2.1.52)). Now in order to derive the quantum Boltzmann equation, we perform the Fourier transform of Eq. (2.2.34) and (2.2.35)

$$\begin{aligned}
& \left[\frac{\partial}{\partial T} + \frac{\vec{p} \cdot \nabla_R}{m} - \nabla_R U_{\text{eff}}(\vec{R}, t) \cdot \nabla_p + \frac{\partial U_{\text{eff}}(\vec{R}, t)}{\partial T} \frac{\partial}{\partial \omega} \right] G'_{11}{}^<(\vec{p}, \omega; \vec{R}, T) \\
&= -i \left[\Sigma_{12}^{HF}(\vec{R}, T) G'_{21}{}^<(\vec{p}, \omega; \vec{R}, T) - \Sigma_{21}^{HF}(\vec{R}, T) G'_{12}{}^<(\vec{p}, \omega; \vec{R}, T) \right] \\
&+ \frac{1}{2} \left[\left(\nabla_R \Sigma_{12}^{HF}(\vec{R}, T) \cdot \nabla_p - \frac{\partial \Sigma_{12}^{HF}(\vec{R}, T)}{\partial T} \frac{\partial}{\partial \omega} \right) G'_{21}{}^<(\vec{p}, \omega; \vec{R}, T) + \right. \\
&+ \left. \left(\nabla_R \Sigma_{21}^{HF}(\vec{R}, T) \cdot \nabla_p - \frac{\partial \Sigma_{21}^{HF}(\vec{R}, T)}{\partial T} \frac{\partial}{\partial \omega} \right) G'_{12}{}^<(\vec{p}, \omega; \vec{R}, T) \right] \\
&- i I_{11}{}^<(\vec{p}, \omega; \vec{R}, T), \tag{2.2.36}
\end{aligned}$$

$$\begin{aligned}
& \left[\frac{\partial}{\partial T} + i2\xi_p(\vec{R}, T) + \frac{\Delta_R}{4m} \right] G'_{12}{}^<(\vec{p}, \omega; \vec{R}, T) \\
&= -i \Sigma_{12}^{HF}(\vec{R}, T) \left[G'_{22}{}^<(\vec{p}, \omega; \vec{R}, T) + G'_{11}{}^<(\vec{p}, \omega; \vec{R}, T) \right] \\
&+ \frac{1}{2} \left(\nabla_R \Sigma_{12}^{HF}(\vec{R}, T) \cdot \nabla_p - \frac{\partial \Sigma_{12}^{HF}(\vec{R}, T)}{\partial T} \frac{\partial}{\partial \omega} \right) (G'_{22}{}^< - G'_{11}{}^<)(\vec{p}, \omega; \vec{R}, T) \\
&- i I_{12}{}^<(\vec{p}, \omega; \vec{R}, T). \tag{2.2.37}
\end{aligned}$$

Here we introduced

$$U_{\text{eff}}(\vec{R}, T) = V_{\text{ext}}(\vec{R}, T) + \Sigma_{11}^{HF}(\vec{R}, t) \tag{2.2.38}$$

as the effective self-consistent Hartree-Fock dynamic mean field, and

$$\xi_p(\vec{R}, T) = \frac{|\vec{p}|^2}{2m} + U_{\text{eff}}(\vec{R}, T) - \mu \tag{2.2.39}$$

as the "normal" single-particle energy. Further the Fourier components of the lesser Green's function appearing in the equations are given by

$$G'_{ij}{}^{<}(\vec{r}, t; \vec{R}, T) = \int \frac{d^3p}{(2\pi)^3} \frac{d\omega}{2\pi} e^{i(\vec{r}\cdot\vec{p}-\omega t)} G'_{ij}{}^{<}(\vec{p}, \omega; \vec{R}, T). \quad (2.2.40)$$

Moreover, the term proportional to the second derivative with respect to \vec{R} can be neglected, since we have assumed before that the Green's functions slowly vary in space and time.

These are the equations of motion for the non-condensate particles. Now we turn our attention to their associated equation for the macroscopic condensate wave function (see Eq. 2.2.17). We are now interested in the integrals of the form

$$\begin{aligned} & \int_{-\infty}^{t_1} d2 [S_{11}^>(1, 2) - S_{11}^<(1, 2)] \Psi_0(2) \\ &= \int_{-\infty}^{t_1} d2 [S_{11}^> - S_{11}^<](\vec{r}_1 - \vec{r}_2, t_1 - t_2; (\vec{r}_1 + \vec{r}_2)/2, (t_1 + t_2)/2) \Psi_0(\vec{r}_2, t_2). \end{aligned} \quad (2.2.41)$$

We assume, that S_{11} is dominated by small values of the relative coordinates $(\vec{r}_1 - \vec{r}_2, t_1 - t_2)$, and therefore we can approximate S_{11}^{\lessgtr} by $S_{11}^{\lessgtr}(\vec{r}_1 - \vec{r}_2, t_1 - t_2; \vec{r}_1, t_1)$. For the same reason, we approximate the macroscopic wave function in Eq. (2.2.41) by $\Psi_0(\vec{r}_2, t_2) \approx \Psi_0(\vec{r}_1, t_1)$. Hence we get (renaming $(\vec{r}_1, t_1) \rightarrow (\vec{R}, T)$)

$$\begin{aligned} & \left[i \frac{\partial}{\partial T} + \frac{1}{2m} \Delta_R - V_{\text{ext}} + \mu - g(n_0(\vec{R}, T) + 2n'(\vec{R}, T)) \right] \Psi_0(\vec{R}, T) \\ &= gm'(\vec{R}, T) \Psi_0^*(\vec{R}, T) + \Psi_0(\vec{R}, T) \int_{-\infty}^T d2 [S_{11}^> - S_{11}^<](\vec{R} - \vec{r}_2, T - t_2; \vec{R}, T) \\ & \quad + \Psi_0^*(\vec{R}, T) \int_{-\infty}^T d2 [S_{12}^> - S_{12}^<](\vec{r}_1 - \vec{r}_2, t_1 - t_2; \vec{r}_1, t_1) \end{aligned} \quad (2.2.42)$$

where the term containing the integrals over S_{12}^{\gtrless} was treated in the same manner as the one containing S_{11}^{\gtrless} . This is the generalized Gross-Pitaevskii equation. The appearance of these two dissipative terms in Eq. (2.2.42) is expected, since the I_{12}^{\gtrless} collisions can change the number of atoms in the condensate and hence modify the magnitude of the condensate macroscopic wave function.

2.2.1 Hartree-Fock approximation and the collisionless Boltzmann Equation

Up to now we have derived the kinetic equations and its associated equation of motion for the macroscopic wave function for a system of bosons at zero temperature. We have included higher order terms, that cause thermalization. In this section we study the collisionless case, and derive for it the quantum Boltzmann equation. For this reason we take the collision integrals I_{11}^{\geq} and I_{12}^{\geq} out of consideration. Furthermore, the Bogoliubov-Hartree-Fock selfenergies, which are the self-energies of interest now, are frequency independent (see [33, 43]). Thus it is sufficient to consider $t = 0$ in Eq. (2.2.34) and Eq. (2.2.35)

$$\begin{aligned} & \left[i \frac{\partial}{\partial T} + \frac{\nabla_{\vec{R}} \cdot \nabla_{\vec{r}}}{m} - \vec{r} \cdot \nabla_{\vec{R}} \left(V_{\text{ext}}(\vec{R}, T) + \Sigma_{11}^{HF}(\vec{R}, T) \right) \right] G'_{11}{}^{<}(\vec{r}; \vec{R}, T) \\ &= \Sigma_{12}^{HF}(\vec{R}, T) G'_{21}{}^{<}(\vec{r}; \vec{R}, T) - \Sigma_{21}^{HF}(\vec{R}, T) G'_{12}{}^{<}(\vec{r}; \vec{R}, T) \\ & \quad + \frac{1}{2} \left[G'_{21}{}^{<}(\vec{r}; \vec{R}, T) \vec{r} \cdot \nabla_{\vec{R}} \Sigma_{12}^{HF}(\vec{R}, T) + G'_{12}{}^{<}(\vec{r}; \vec{R}, T) \vec{r} \cdot \nabla_{\vec{R}} \Sigma_{21}^{HF}(\vec{R}, T) \right], \end{aligned} \quad (2.2.43)$$

$$\begin{aligned} & \left[i \frac{\partial}{\partial T} + \frac{\Delta_{\vec{R}}}{4m} + \frac{\Delta_{\vec{r}}}{m} - 2V_{\text{ext}}(\vec{R}, T) - 2\Sigma_{11}^{HF}(\vec{R}, t) \right] G'_{12}{}^{<}(\vec{r}; \vec{R}, T) \\ &= \Sigma_{12}^{HF}(\vec{R}, T) G'_{22}{}^{<}(\vec{r}; \vec{R}, T) + \Sigma_{12}^{HF}(\vec{R}, T) G'_{11}{}^{<}(\vec{r}; \vec{R}, T) \\ & \quad + \frac{1}{2} \left[G'_{22}{}^{<}(\vec{r}; \vec{R}, T) \vec{r} \cdot \nabla_{\vec{R}} \Sigma_{12}^{HF}(\vec{R}, T) - G'_{11}{}^{<}(\vec{r}; \vec{R}, T) \vec{r} \cdot \nabla_{\vec{R}} \Sigma_{12}^{HF}(\vec{R}, T) \right]. \end{aligned} \quad (2.2.44)$$

Then we make use of the fact that the lesser Green's functions are connected with the single-particle distribution function $f_1(\vec{p}, \vec{R}, T)$. We therefore define

$$f_1(\vec{p}, \vec{R}, T) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} i G'_{11}{}^{<}(\vec{p}, \omega; \vec{R}, T). \quad (2.2.45)$$

By definition, the non-condensate particles density is given by

$$n'(\vec{R}, T) = \int \frac{d^3p}{(2\pi)^3} f_1(\vec{p}, \vec{R}, T). \quad (2.2.46)$$

We can see that $f_1(\vec{p}, \vec{R}, T)$ corresponds to the well known Wigner distribution function. Due to the finite value of the order parameter the off-diagonal elements of the single-boson propagator do not vanish. Since the anomalous Green's functions have a non zero value, it is natural to introduce an additional distribution for the non-condensate atoms which will give us the anomalous non-condensate particles density $m'(\vec{R}, T)$, namely

$$f_2(\vec{p}, \vec{R}, T) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} i G'_{12}{}^{<}(\vec{p}, \omega; \vec{R}, T) \quad (2.2.47)$$

with the corresponding anomalous density

$$m'(\vec{R}, T) = \int \frac{d^3p}{(2\pi)^3} f_2(\vec{p}, \vec{R}, T). \quad (2.2.48)$$

It is important to mention that, as defined, the distribution functions f_1 and f_2 describe the behavior of the particles. They should not be confused with the distribution function for the quasiparticle excitations.

Now, taking the Fourier transform of Eq. (2.2.43) and Eq. (2.2.44) and using the definition of the distribution function, we obtain

$$\begin{aligned} & \left[\frac{\partial}{\partial T} + \frac{\vec{p} \cdot \nabla_R}{m} - \nabla_R U_{\text{eff}}(\vec{R}, T) \cdot \nabla_p \right] f_1(\vec{p}, \vec{R}, T) \\ &= -i \left[\Sigma_{12}^{HF}(\vec{R}, T) f_2^*(-\vec{p}, \vec{R}, T) - \Sigma_{21}^{HF}(\vec{R}, T) f_2(\vec{p}, \vec{R}, T) \right] \\ & \quad + \frac{1}{2} \left[\nabla_R \Sigma_{12}^{HF}(\vec{R}, T) \cdot \nabla_p f_2^*(-\vec{p}, \vec{R}, T) + \nabla_R \Sigma_{21}^{HF}(\vec{R}, T) \cdot \nabla_p f_2(\vec{p}, \vec{R}, T) \right], \end{aligned} \quad (2.2.49)$$

$$\begin{aligned} & \left[\frac{\partial}{\partial T} + i2\xi_p(\vec{R}, T) \right] f_2(\vec{p}, \vec{R}, T) \\ &= -i\Sigma_{12}^{HF}(\vec{R}, T) \left[f_1(\vec{p}, \vec{R}, T) + f_1(-\vec{p}, \vec{R}, T) + 1 \right] \\ & \quad + \frac{1}{2} \nabla_R \Sigma_{12}^{HF}(\vec{R}, T) \cdot \nabla_p \left(f_1(\vec{p}, \vec{R}, T) - f_1(-\vec{p}, \vec{R}, T) \right), \end{aligned} \quad (2.2.50)$$

where U_{eff} and $\xi_p(\vec{R}, T)$ were defined in Eq. (2.2.38) and Eq. (2.2.39), respectively. These are the Bogoliubov-Hartree-Fock collisionless kinetic equations for the case that the Green's function vary slowly in time and space.

On the other hand, the equation of motion for the order parameter is given by

$$\left[i \frac{\partial}{\partial T} + \frac{1}{2m} \Delta_R - V_{\text{ext}} + \mu - g(n_0 + 2n')(\vec{R}, T) \right] \Psi_0(\vec{R}, T) = gm'(\vec{R}, T) \Psi_0^*(\vec{R}, T). \quad (2.2.51)$$

This equation is exact within the Bogoliubov-Hartree-Fock approximation, i.e. it is not limited to slowly varying disturbances.

Chapter 3

Quantum Coherent Particle Tunneling between Two Trapped Bose-Einstein Condensates

Manipulations with two trapped condensates provide a fascinating possibility to observe new quantum phenomena on macroscopic scales, related to the superfluid phase of the condensate. A far off-resonant intense laser sheet divided a trapped condensate, creating a high barrier in between. Switching off the double well trap, the two released condensates overlapped, producing a robust *two-slit* atomic interference pattern, clear signature of phase coherence over a macroscopic scale. The non-destructive detection of phase differences between two trapped Bose-Einstein condensates could be achieved by lowering the intensity of the laser sheet [15]. This allows atomic tunneling through the barrier, and the detection of Josephson-like current-phase effects, which are going to be the subject of study of this chapter.

In the preceding chapters we presented the theory for a system consisting of bosons at zero temperature and its non-equilibrium extension. Before we can study the non-adiabatic mixing of two trapped Bose-Einstein condensates, making use of these two techniques, we have to understand what happens in the *meanfield regime*, when we replace the boson field operator by its expectation value.

3.1 Two-Mode Approximation

We start with a system consisting of two Bose gases confined in a double well potential at zero temperature (see Fig 3.1). A coherent oscillating particle tunneling between the wells is induced. This can be understood as a macroscopic superposition of all possible particle states. As we have shown in Section 1.1 the particle number (or density) and the macroscopic phase are canonically conjugated. If we start an experiment with two condensates that are independent from each other, each of them would have a definite particle number and their macroscopic phase would be

a superposition of all phases between 0 to 2π . Then after we link the condensates, a particle flux between the condensates sets in and we can measure the relative phase difference between the condensates. The dynamics of such a system is well described by two coupled Gross-Pitaevskii equations for the condensate amplitudes. The coupling element is a transfer matrix, which we identify with *Josephson Coupling*.

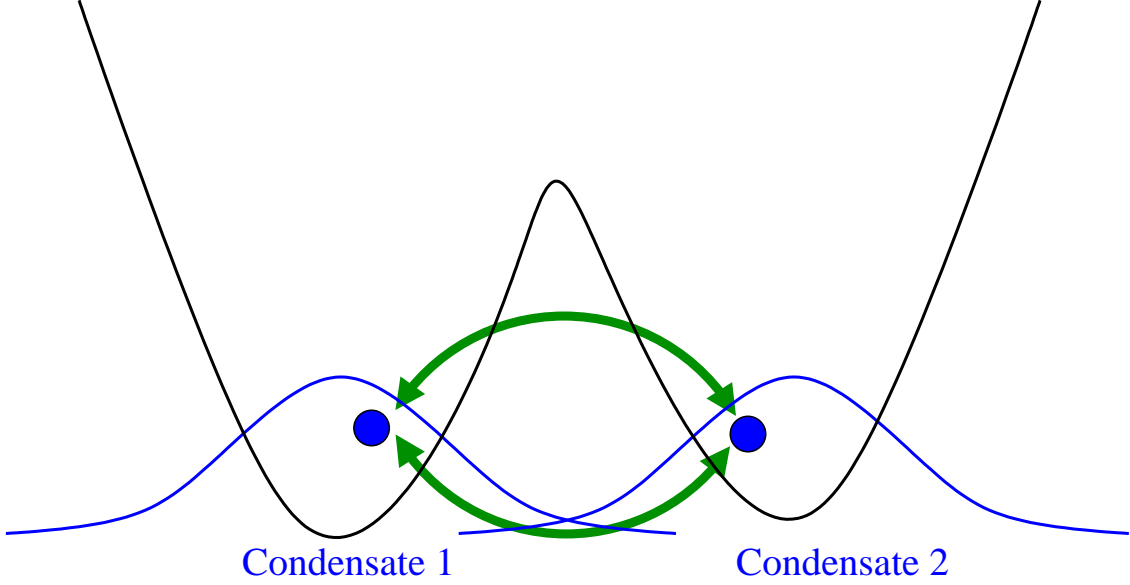


Figure 3.1: The double well trap with the two Bose-Einstein condensates and the Josephson tunneling between them. Here we only display the linear particle tunneling

In order to derive the equations of motion we consider the Hamiltonian (1.0.3), which describes a weakly interacting Bose gas and replace the operator Ψ by its expectation value $\Psi_0 = \langle \Psi(\vec{x}, t) \rangle$. Further we assume that the barrier separating the condensates is sufficiently high such that the number of particles in it is exponentially suppressed. We can thus decompose the condensate amplitude Ψ_0 into two orthogonal modes, describing each condensate:

$$\Psi_0(\vec{x}, t) = a_1(t)\phi_1(\vec{x}) + a_2(t)\phi_2(\vec{x}) \quad (3.1.1)$$

where ϕ_1 and ϕ_2 are the solutions of the time-independent Gross-Pitaevskii equations for each well and are assumed to be real. This decomposition is called the *two-mode approximation* (see also [27, 31, 30]). Now plugging this decomposition in the Hamiltonian (1.0.3) we obtain

$$\begin{aligned} H = & (E - \mu)(a_1^*a_1 + a_2^*a_2) - \frac{J}{2}(a_1^*a_2 + a_2^*a_1) + \\ & + \frac{T_0}{2}(a_1^*a_1a_1^*a_1 + a_2^*a_2a_2^*a_2) + 2T_2a_1^*a_1a_2^*a_2 + \frac{T_2}{2}(a_1^*a_2a_1^*a_2 + a_2^*a_1a_2^*a_1) \\ & + T_1(a_1^*a_1a_1^*a_2 + a_1^*a_1a_2^*a_1 + a_2^*a_2a_2^*a_1 + a_2^*a_2a_1^*a_2) \end{aligned} \quad (3.1.2)$$

with the following parameters

$$E = \int d^3x \phi_i \left(-\frac{1}{2m} \Delta + V_{\text{ext}}(x, t) \right) \phi_i, \quad \text{for } i = 1, 2 \quad (3.1.3)$$

as the lowest state of energy¹,

$$J = -2 \int d^3x \phi_j \left(-\frac{1}{2m} \Delta + V_{\text{ext}}(x, t) \right) \phi_i, \quad \text{for } i \neq j = 1, 2 \quad (3.1.4)$$

as the Josephson coupling element,

$$T_1 = g \int d^3x (\phi_1)^3 \phi_2 = g \int d^3x (\phi_2)^3 \phi_1 \quad (3.1.5)$$

and

$$T_2 = g \int d^3x (\phi_1)^2 (\phi_2)^2 \quad (3.1.6)$$

as overlap integrals, and

$$T_0 = g \int d^3x (\phi_1)^4 = g \int d^3x (\phi_2)^4 \quad (3.1.7)$$

as the integral corresponding to the intrawell interatomic interactions. After some algebra we can rewrite the expression (3.1.2) as follows

$$\begin{aligned} H = & (E - \mu)(a_1^* a_1 + a_2^* a_2) - \frac{J}{2}(a_1^* a_2 + a_2^* a_1) + \\ & + \frac{T_0}{4}(a_1^* a_1 + a_2^* a_2)^2 + \frac{T_0}{4}(a_1^* a_1 - a_2^* a_2)^2 + T_2 a_1^* a_1 a_2^* a_2 + \frac{T_2}{2}(a_1^* a_2 + a_2^* a_1)^2 \\ & + T_1(a_1^* a_1 + a_2^* a_2) \cdot (a_2^* a_1 + a_1^* a_2). \end{aligned} \quad (3.1.8)$$

We have thus

$$\begin{aligned} H = & (E - \mu)(a_1^* a_1 + a_2^* a_2) - \left[\frac{J}{2} - T_1(a_1^* a_1 + a_2^* a_2) \right] (a_1^* a_2 + a_2^* a_1) + \\ & + \frac{T_2}{2}(a_1^* a_2 + a_2^* a_1)^2 + \frac{T_0 + 2T_2}{4}(a_1^* a_1 + a_2^* a_2)^2 + \frac{T_0 - 2T_2}{4}(a_1^* a_1 - a_2^* a_2)^2. \end{aligned} \quad (3.1.9)$$

In Section 1.1 we showed that the condensate amplitude can be written in terms of the particle number and the macroscopic phase (see Eq. (1.1.6)). We now make

¹The lowest state of energy E is equal for both wells, since we are considering a symmetric trap

use of this fact and replace a_j by $\sqrt{N_j}e^{i\theta_j}$ in the Hamiltonian (3.1.9), so that we obtain

$$H = (E - \mu)N_0 + \frac{T_0 + 2T_2}{4}N_0^2 - [J - 2T_1N_0]\sqrt{N_0^2 - z^2}\cos\theta + 2T_2\cos^2\theta + \frac{T_0 - 2T_2}{4}z^2 \quad (3.1.10)$$

where we introduce the total particle number

$$N_0 = N_1 + N_2, \quad (3.1.11)$$

the particle imbalance

$$z = N_1 - N_2 \quad (3.1.12)$$

between the two condensates and the relative phase difference

$$\theta = \theta_2 - \theta_1. \quad (3.1.13)$$

Notice that the total particle number N_0 is constant. In the following we assume, that:

- lowest energy E is equal to μ .
- although N_0 is a huge number, the term proportional to $2T_1N_0$ can be neglected, since T_1 is very small and considered values for the Josephson coupling J are much larger than T_1N_0 .
- we take the nonlinear tunneling represented by the term proportional to $\cos^2\theta$ out of consideration, since it displays the same physics as the linear one.
- since N_0 is a conserved quantity, we can get rid off the remaining terms containing N_0 by performing a rotation of the a 's.
- T_2 is negligibly small with respect to T_0 .

After these simplifications we end up with the following simple Hamiltonian, which describes a canonical Josephson junction

$$H = -J\sqrt{N_0^2 - z^2}\cos\theta + \frac{T_0}{4}z^2. \quad (3.1.14)$$

For convenience, we express the energy in terms of the total particle number N_0 and the Josephson coupling J :

$$H \equiv \left(\frac{H}{JN_0}\right) = \sqrt{1 - z^2}\cos\theta + \frac{U}{2}z^2, \quad (3.1.15)$$

where the particle imbalance z has been renormalized to the total particle number

$$z \mapsto z = \frac{N_1 - N_2}{N_0} \quad (3.1.16)$$

and U is defined as

$$U = \frac{T_0 N_0}{2J}. \quad (3.1.17)$$

We observe, that the normalized particle imbalance z and the relative phase difference θ are canonically conjugated. The equations of motion can be computed straightforward from

$$\dot{z} = -\frac{\partial H}{\partial \theta} \quad \text{and} \quad \dot{\theta} = \frac{\partial H}{\partial z} \quad (3.1.18)$$

and the result is (see also [27])

$$\dot{z}(t) = -\sqrt{1 - z^2(t)} \sin \theta(t), \quad (3.1.19)$$

$$\dot{\theta}(t) = Uz + \frac{z(t)}{\sqrt{1 - z^2(t)}} \cos \theta(t). \quad (3.1.20)$$

In a simple mechanical analogy, this system of coupled differential equations describes a *nonrigid* pendulum, of tilt angle θ and length proportional to $\sqrt{1 - z^2}$, that decreases with the "angular momentum" z .

The Bose Josephson junction intertrap tunneling is given by

$$I = \dot{z}N_0 = J\sqrt{1 - z^2} \sin \theta \quad (3.1.21)$$

with J as the Josephson coupling. Here we notice that Eq. (3.1.21) differs from Cooper-pair superconductor Josephson junction tunneling current in its nonlinearity in z . Thus the equations for $\dot{\theta}$ also differ. The detailed analysis of Eqs. (3.1.19) and (3.1.20) with exact analytical solutions in terms of Jacobian and Weierstrassian elliptic functions is presented in [29]. Here we consider three different regimes.

1. Noninteracting limit

In this limit we consider negligibly small interatomic interaction ($U \rightarrow 0$). Rabi-like oscillations in the population of each trap with the frequency

$$\omega_R = J \quad (3.1.22)$$

are displayed by Eqs. (3.1.19) and (3.1.20). These oscillations are equivalent to a single-atom dynamics, rather than a Josephson effect arising from the interacting superfluid condensate.

2. Linear regime

In this regime we consider small amplitude oscillations, i.e.

$$|z| \ll 1 \quad |\theta| \ll 1. \quad (3.1.23)$$

The Eqs. (3.1.19) and (3.1.20) hence simplify as follows

$$\dot{z} \simeq -\theta \quad (3.1.24)$$

$$\dot{\theta} \simeq (U + 1)z. \quad (3.1.25)$$

These set of coupled differential equations describe the small amplitude oscillations of the pendulum analog, with a sinusoidal $z(t)$ with the frequency

$$\omega_L = \sqrt{\frac{T_0 N_0 J}{2} + J^2}. \quad (3.1.26)$$

Here one should keep in mind that the energies above were given in terms of the Josephson coupling J . Moreover, we observe that the Bose Josephson junction oscillations of the population should show up as temporal oscillations of phase-contrast patterns.

Further, in order to justify the neglect of the spatial variations of z and θ we assume that the Josephson-like length

$$\lambda_J \equiv \frac{1}{\sqrt{2mJ}} \quad (3.1.27)$$

which governs the spatial variation along the junction, should be much larger than the length characterized by the junction area.

3. Nonlinear regime

Numerical solutions of the Eqs. (3.1.19) and (3.1.20) yield a non-sinusoidal behavior of the particle imbalance, that can be understood as the anharmonic *generalization* of the sinusoidal Josephson effects. This anharmonicity can be achieved by increasing the initial particle imbalance $z(0)$ for a fixed interatomic interaction U (or increasing U for a fixed $z(0)$). This behavior corresponds to the large amplitude oscillations of the nonrigid pendulum. In Fig. (3.2) we display how the anharmonic behavior increases with U for a fixed $z(0)$.

In addition to anharmonic and critical slow oscillations other striking effects occur in Bose Josephson junction. For instance, for a fixed value of the initial particle imbalance $z(0)$, if the selfinteraction parameter U exceeds a critical value U_c , the populations become macroscopically *selftrapped* with

$$\langle z(t) \rangle \neq 0. \quad (3.1.28)$$

This phenomenon can be understood through the pendulum analogy. If the population imbalances are prepared such that the initial angular kinetic energy $z(0)^2$ exceeds the potential barrier height of the vertically displaced $\theta = \pi$ "pendulum orientation", a steady self sustained pendulum rotation will occur,

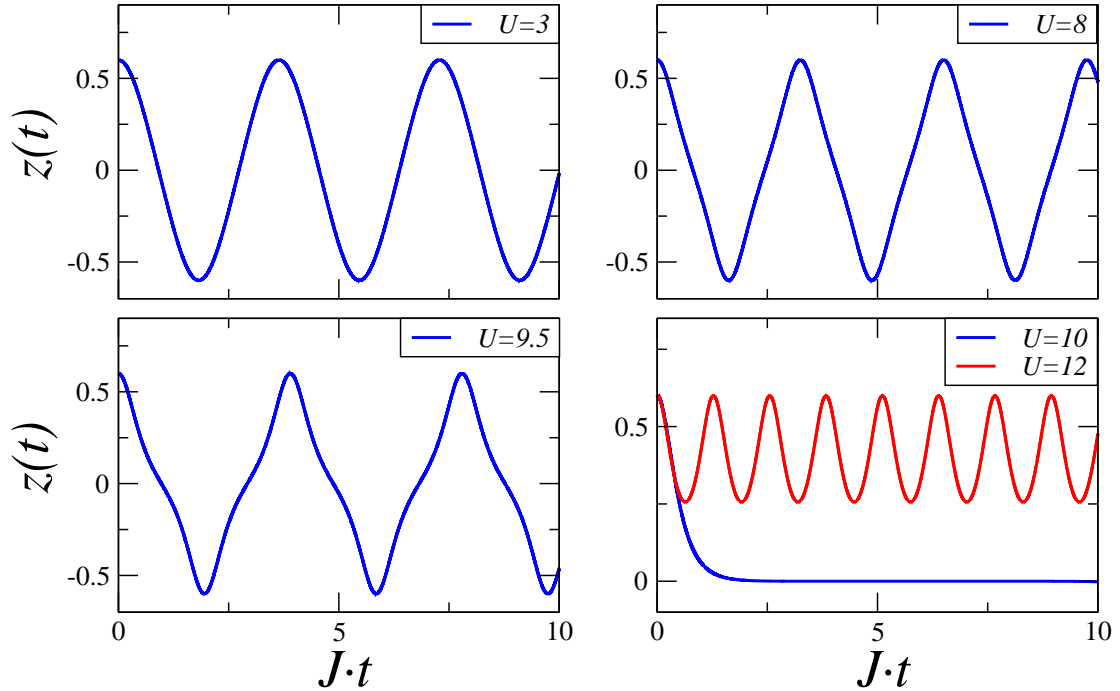


Figure 3.2: $z(t)$ as a function of $J \cdot t$ with initial conditions $z(0) = 0.6$ and $\theta(0) = 0$ in a symmetric trap. The red line at the bottom displays the selftrapping behavior.

with nonzero angular momentum $\langle z \rangle$ and a closed loop trajectory around the pendulum support.

This analogy will help us to formulate an expression for the critical value of the selfinteractions parameter U_c . For this purpose we make use of the fact, that the total energy of the system is constant and observe that the system can only be macroscopically selftrapped if its energy is larger than the hopping energy, i.e. larger than the energy corresponding to the Josephson junction effect (see also [27]):

$$H_0 \equiv H(z(0), \theta(0)) = \sqrt{1 - z(0)^2} \cos(\theta(0)) + \frac{U}{2} z(0)^2 > 1. \quad (3.1.29)$$

Here the energy is given in terms of the Josephson coupling J . In this manner we can equivalently formulate the condition for macroscopically selftrapping in terms of the selfinteractions parameter:

$$U > U_c = \frac{1 + \sqrt{1 - z(0)^2} \cos(\theta(0))}{z(0)^2/2}. \quad (3.1.30)$$

The macroscopic selftrapping for increasing U and fixed $z(0)$ is shown in Fig. (3.2). The experimental observation of this behavior involves a series

of experiments in which $\theta(0)$ and $z(0)$ are kept constant but U is varied by changing the geometry or the total number of condensate atoms, for example [15, 14].

On the other hand, changing the initial value of the population imbalance $z(0)$ with a fixed trap geometry, total number of condensate atoms and initial relative phase difference $\theta(0)$, U remains constant and Eq. (3.1.29) defines a critical initial population imbalance z_c . For $\theta(0) = 0$, if $z(0) < z_c$, macroscopic quantum self-trapping sets in. For $\theta(0) = \pi$ the dynamics are presented

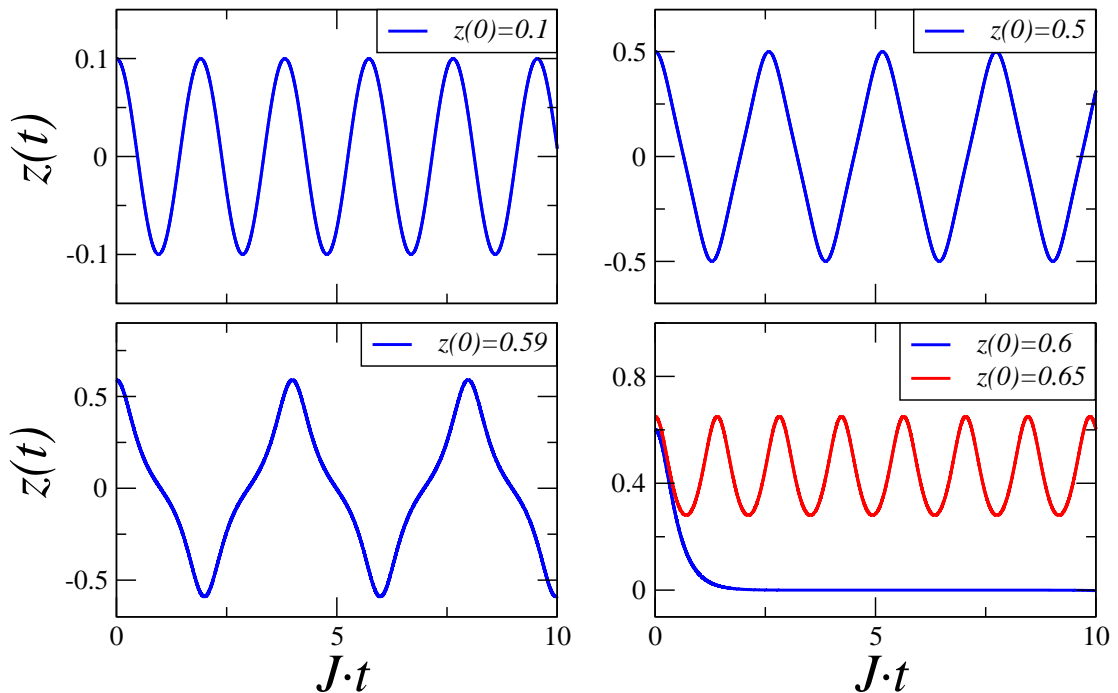


Figure 3.3: $z(t)$ as a function of Jt for a fixed selfinteraction parameter $U = 10$ and $\theta(0) = 0$ in a symmetric trap. The red line at the bottom displays the selftrapping behavior.

The dynamical behavior of the Bose Josephson junction system can be summarized quite conveniently in terms of a phase portrait of the two dynamical variables z and θ , as shown in Fig. 3.4. At the top the trajectories are calculated for different selfinteraction parameter U with $z(0)$ kept constant at 0.6. The closed lines (black, dark green and maroon) display the evolution of the system, where the phase θ and particle imbalance z oscillate around 0. This corresponds to the case when the system is governed by the Josephson energy and, hence, one observes

Bose Josephson oscillations. Then increasing U (by changing the geometry of the trap or the total number of particles) the system undergoes a transition to the selftrapping regime. The blue line shows the critical behavior and the red line displays a macroscopically selftrapped system. The figure at the bottom displays a

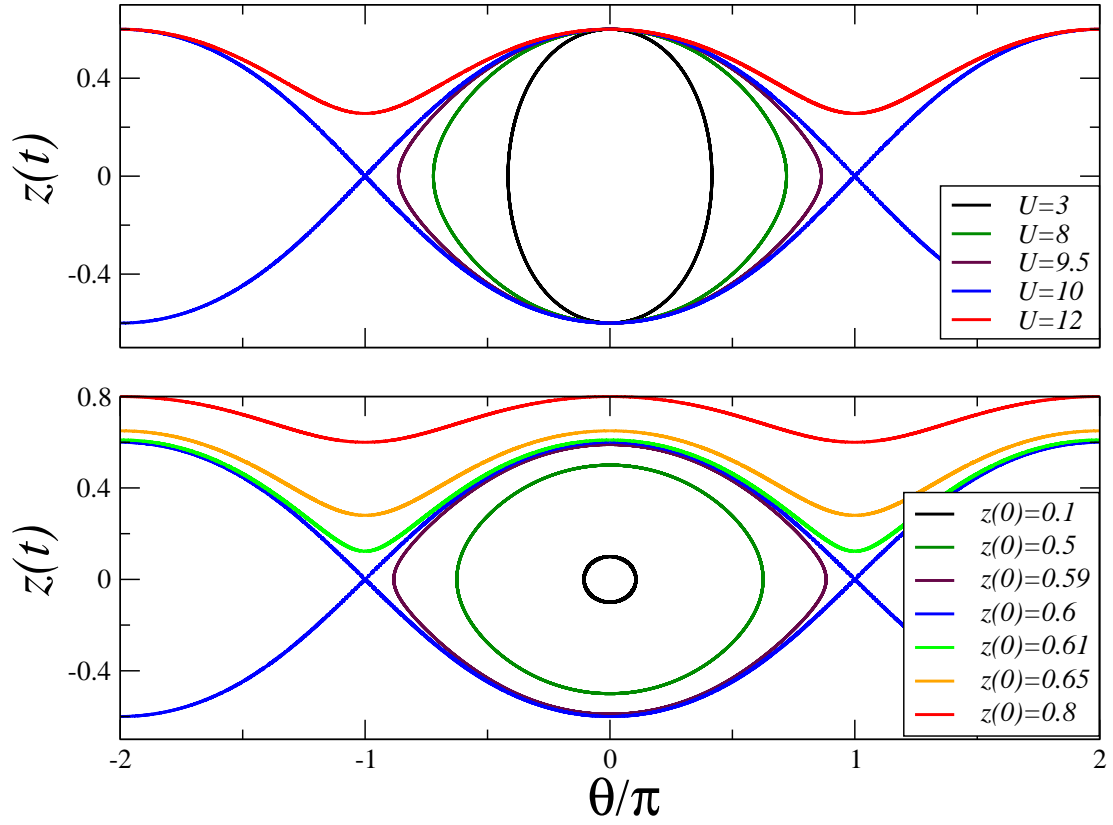


Figure 3.4: Phases-space portrait of the dynamical variables z and θ for fixed initial particle imbalance $z(0) = 0.6$ and different selfinteraction parameter U at the top and for fixed $U = 10$ and different $z(0)$ at the bottom.

Chapter 4

Non-Adiabatic Mixing of trapped Bose Gases at Zero Temperature

In chapter 3 we presented the two-mode model for the description of the coherent particle tunneling between two trapped Bose-Einstein condensates. The theory presented allows the investigation of the two different dynamical phenomena known for a system consisting of two Bose-Einstein condensates confined in a double well trap at zero temperature - Josephson oscillations and selftrapping. We considered the limit of a large number of condensed particles in the *semiclassical* or *mean field* approximation. In this chapter we will include the single particle excitations, which will deliver the corrections to the set of coupled Gross-Pitaevskii equations. Moreover we assume that the linking of the two trapped Bose gases happens in a non-adiabatic way and evokes therefore nonequilibrium features, that have to be analyzed with the tools presented in chapter 2.1.

4.1 Microscopic derivation of the Hamiltonian

Consider the most general Hamiltonian (1.0.3) describing a dilute Bose gas. Now we consider two Bose gases confined in a double well potential V_{ext} at zero temperature (see Fig. (3.1)). As we have done before, we assume that the barrier in the middle separating the two gases is high enough, so that the number of particles in it is exponentially suppressed. We can therefore decompose, like in Chapter 3, the order parameter into two modes that are orthogonal to each other. Moreover, since we want to include the corrections to the Gross-Pitaevskii equations, we include the single particle excitations of the system. Hence the bosonic field operator becomes

$$\Psi(\vec{x}, t) = \underbrace{\Psi_1(\vec{x}, t) + \Psi_2(\vec{x}, t)}_{\Psi_0(\vec{x}, t)} + \Psi'(\vec{x}, t) \quad (4.1.1)$$

where $\Psi_0(\vec{x}, t) = \langle \Psi(\vec{x}, t) \rangle$ is the order parameter, which is decomposed in two orthogonal modes, Ψ_1 and Ψ_2 corresponding to the two condensates, and Ψ' is the

bosonic field operator corresponding to the single particle excitations. Here Ψ' is not decomposed in two states orthogonal to each other, because we have assumed, that the energy corresponding to the lowest single particle excitation state of each well is higher than the barrier in the middle separating the two gases. It is therefore a good approximation to consider "only one spectrum" of single particle excitations corresponding to both wells. One of the possible experimental realizations of a double well confining trap involves the combination of a three dimensional harmonic confinement and a one dimensional periodic potential with a large lattice spacing [15]. We can therefore take the spectrum of the single particle excitations equal to the one corresponding to the harmonic confinement. This idea is presented in Fig. 4.1.

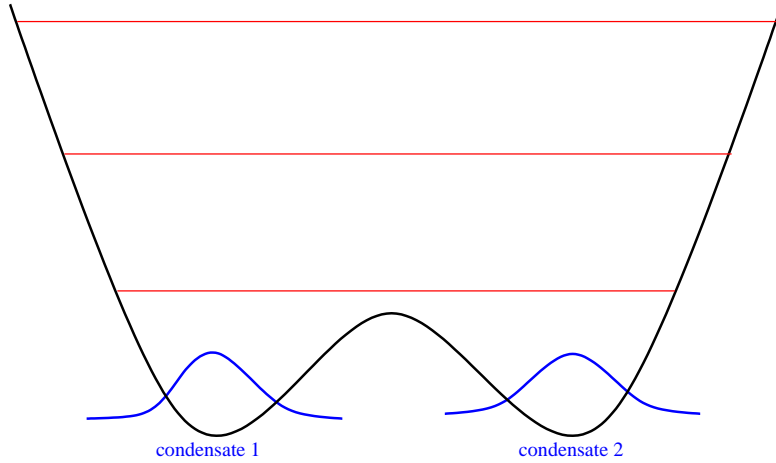


Figure 4.1: Double well potential as a result of a combination of a three dimensional harmonic confinement and a one dimensional periodic potential with large lattice spacing. The energy spectrum showed belongs to harmonic confinement and is assumed to coincide with the spectrum of the single particle excitations of both wells

Now we proceed in similar way like in chapter 3 and separate the space dependence from the time dependence

$$\Psi(\vec{x}, t) = \phi_1(\vec{x})a_1(t) + \phi_2(\vec{x})a_2(t) + \sum_{n \neq 0} \varphi_n(\vec{x})b_n(t) \quad (4.1.2)$$

and

$$\Psi^\dagger(\vec{x}, t) = \phi_1(\vec{x})a_1^*(t) + \phi_2(\vec{x})a_2^*(t) + \sum_{n \neq 0} \varphi_n(\vec{x})b_n^\dagger(t), \quad (4.1.3)$$

where ϕ_1 , ϕ_2 and φ_n are assumed to be real, a_1 and a_2 are complex functions and b_n are bosonic field operators corresponding to the one particles excitations.

Now making use of the ansatz's (4.1.2) and (4.1.3) and plugging them in the Hamiltonian (1.0.3) we obtain

$$H = H_{aa} + H_{ab} \quad (4.1.4)$$

where H_{aa} is the Hamiltonian (3.1.9) computed in Chapter 3 in the "semiclassical" two-mode approximation and H_{ab} contains all the terms involving the operators b_n corresponding to the single particle excitation.

Now we turn our attention to H_{ab} and compute it explicitly. Terms containing only one particle operator b_n or b_n^\dagger corresponding to the single particle do not enter the Hamiltonian, as we have seen in section 1.2. We also omitted the terms involving three b_n 's and one macroscopic wave function. They may be included in order to study thermalization effects. This kind of terms appears as second order processes and can be understood as collision terms. By retaining all the quadratic and quartic terms in b_n , we end up with the following Hamiltonian

$$\begin{aligned} H_{ab} = & \sum_{n,m} (\epsilon_{nm} - \mu + C_{nm}N_0 + D_{nm}(a_1^*a_2 + a_2^*a_1)) b_n^\dagger b_m + \\ & + \sum_{i=1}^2 \sum_{n,m} \left(\frac{C_{nm}}{4} a_i^* a_i^* b_n b_m + \text{h.c.} \right) + \sum_{n,m} \left(\frac{D_{nm}}{2} a_1^* a_2^* b_n b_m + \text{h.c.} \right) + \\ & + \sum_{n,m} \sum_{n',m'} B_{nmn'm'} b_n^\dagger b_m^\dagger b_{n'} b_{m'}, \end{aligned} \quad (4.1.5)$$

where

$$\epsilon_{nm} = \int d^3x \varphi_n \left(-\frac{1}{2m} \Delta + V_{\text{ext}}(x, t) \right) \varphi_m \quad (4.1.6)$$

is the energy spectrum of the single particle excitations,

$$C_{nm} = 2g \int d^3x \varphi_n \varphi_m |\phi_i|^2 \quad \text{for } i = 1, 2 \quad (4.1.7)$$

and

$$D_{nm} = 2g \int d^3x \varphi_n \varphi_m \phi_1 \phi_2 \quad (4.1.8)$$

are overlap integrals, and

$$B_{nmn'm'} = g \int d^3x \varphi_n \varphi_m \varphi_{n'} \varphi_{m'} \quad (4.1.9)$$

is the integral corresponding to the interactions of the particles outside of the condensate.

Now, in order to simplify our Hamiltonian, we perform some approximations:

- As it was done in chapter 3, we take the nonlinear tunneling expressed by the term proportional to $(a_1^*a_2 + a_2^*a_1)^2$ in the Hamiltonian H_{aa} out of consideration, since it displays the same physics as the linear tunneling and is less probable,
- term proportional to $T_1(a_1^*a_1 + a_2^*a_2)$ is negligibly small with respect to the values of the Josephson coupling J under consideration,
- T_2 is negligibly small with respect to T_0 ,
- we assume that the creation and the annihilation of two non-condensate particles described by terms like $a_i a_j b_n^\dagger b_m^\dagger$ and $a_i^* a_j^* b_n b_m$ is less probable for two particles coming from different levels of being created in different levels. In other words, we can make the following replacements

$$\epsilon_{nm} \approx \epsilon_n \delta_{nm}, \quad C_{nm} \approx C \delta_{nm} \quad \text{and} \quad D_{nm} \approx D \delta_{nm}, \quad (4.1.10)$$

where ϵ_n , C and D are real numbers. This is a suitable approximation, since $\{\varphi\}$ is a set of orthonormal functions,

- at last, we treat the interaction term involving four b_n 's in Hartree-Fock approximation, i.e.

$$\sum_{n,m} \sum_{n',m'} B_{nmn'm'} b_n^\dagger b_m^\dagger b_{n'} b_{m'} = B \sum_{n,m} \left[2 \langle b_n^\dagger b_n \rangle b_m^\dagger b_m + \frac{1}{2} (\langle b_n^\dagger b_n^\dagger \rangle b_m b_m + \text{h.c.}) \right] \quad (4.1.11)$$

where $\langle b_n^\dagger b_n^\dagger \rangle$ and $\langle b_n b_n \rangle$ are nonzero due to the finite value of the order parameter [20].

In the preceding chapter we were able to perform a rotation, in order to get rid off the term proportional to the square of the number of particles of both condensates. In the "semiclassical" approximation the condensate number of particles is a conserved quantity. This is not the fact for the system under consideration. The total number of particles of the system is given by

$$N_{\text{tot}} = a_1^*(t)a_1(t) + a_2^*(t)a_2(t) + \langle b_n^\dagger(t+)b_n(t) \rangle, \quad (4.1.12)$$

where the notation $t+$ means that the time argument of b_n^\dagger is infinitesimally larger than the time argument of the other b_n . Since $N_0 = N_1 + N_2$ is not a conserved quantity anymore, we have to keep the term proportional to N_0^2 .

Inserting all the simplifications listed above in the Hamiltonian (4.1.4) we obtain

$$\begin{aligned}
H = & -\frac{J}{2}(a_1^*a_2 + a_2^*a_1) + \left[(E - \mu)N_0 + \frac{T_0}{4}N_0^2 \right] + \frac{T_0}{4}(a_1^*a_1 - a_2^*a_2)^2 + \\
& + \sum_n (\epsilon_n - \mu + CN_0 + D(a_1^*a_2 + a_2^*a_1)) b_n^\dagger b_n + \\
& + \frac{C}{4} \sum_{i=1}^2 \sum_n (a_i^* a_i^* b_n b_n + \text{h.c.}) + \frac{D}{2} \sum_n (a_1^* a_2^* b_n b_n + \text{h.c.}) + \\
& + B \sum_{n,m} \left[2 \langle b_n^\dagger b_n \rangle b_m^\dagger b_m + \frac{1}{2} (\langle b_n^\dagger b_n^\dagger \rangle b_m b_m + \text{h.c.}) \right]. \tag{4.1.13}
\end{aligned}$$

The first line of the Hamiltonian displays the dynamics of both condensates. The first term with the coupling $J/2$ describes the coherent particle tunneling between the two condensates, the two terms involving four macroscopic wave functions belong to the interatomic interactions of the condensates, and EN_0 is the kinetic term. The other lines describe the behavior of the single particle excitations and how they are coupled to the condensates. The terms proportional to D exhibit a kind of "auxiliary bypass" from one well to the other¹ and the terms involving the coupling C are responsible for the "intrawell" coupling between the condensate and the particles outside of the condensate. Some of these processes are illustrated in Fig. (4.2).

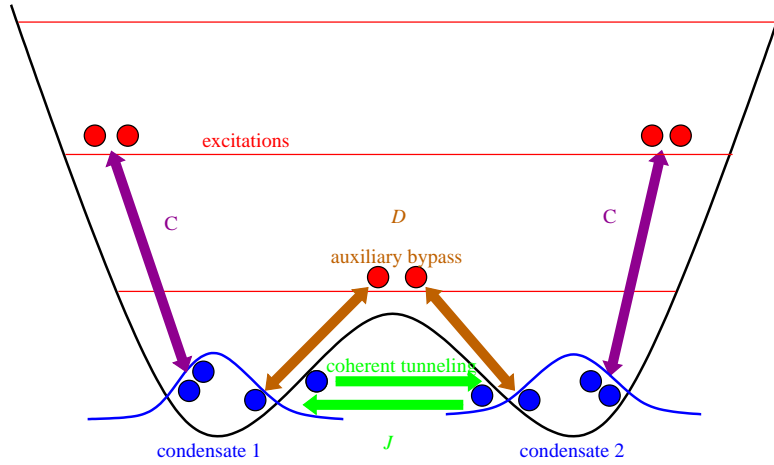


Figure 4.2: Here some of the processes displayed by the Hamiltonian (4.1.13).

In the rest of this chapter we will focus on this Hamiltonian and derive from it a system of selfconsistent equations consisting of the kinetic equations for the

¹In this thesis we call "auxiliary bypass" to the bypass consisting of excitations, over which particles maybe transferred from one well to the other

particle outside of the condensate and associated equations for both macroscopic wave functions. The tools used for the derivation were presented in Chapter 2.1 and 1.

4.2 Kinetic Equations

Above we presented the model we want to consider, and derived microscopically the Hamiltonian that describes the situation. Further, in chapter 2 we present the non-equilibrium extension for a system of bosons at zero temperature. In the present section we want to make use of this formalism in order to derive the kinetic equations for our model.

We now start computing the equations of motion for the field operators b_n^\dagger and b_n corresponding to the creation and annihilation operator of the non-condensate particles:

$$\begin{aligned} & \left[i \frac{\partial}{\partial t} - \epsilon_n + \mu - \left(CN_0 + D(a_1^* a_2 + a_2^* a_1) + 2B \sum_m \langle b_m^\dagger b_m \rangle \right) \right] b_n(t) \\ &= \left(\frac{C}{2} \sum_{i=1}^2 a_i a_i + D a_1 a_2 + B \sum_m \langle b_m b_m \rangle \right) b_n^\dagger(t), \end{aligned} \quad (4.2.1)$$

$$\begin{aligned} & \left[-i \frac{\partial}{\partial t} - \epsilon_n + \mu - \left(CN_0 + D(a_1^* a_2 + a_2^* a_1) + 2B \sum_m \langle b_m^\dagger b_m \rangle \right) \right] b_n^\dagger(t) \\ &= \left(\frac{C}{2} \sum_{i=1}^2 a_i^* a_i^* + D a_1^* a_2^* + B \sum_m \langle b_m^\dagger b_m^\dagger \rangle \right) b_n(t). \end{aligned} \quad (4.2.2)$$

It is not a difficult task to compute from these equations the equations of motion for the Green's functions. The procedure may be found in the literature [34, 33, 18]. After some algebra we find for the matrix representation of the non-condensate particle Green's function in Bogoliubov space in real time

$$\begin{aligned} & \sum_m \int d\bar{t} \left[(\mathbf{G}_0)_{nm}^{-1}(t, \bar{t}) - \Sigma_{nm}^{HF}(t, \bar{t}) \right] \mathbf{G}'_{ml}^{\geq}(\bar{t}, t') \\ &= \sum_m \int_{-\infty}^t d\bar{t} \mathbf{\Gamma}_{nm}(t, \bar{t}) \mathbf{G}'_{ml}^{\geq}(\bar{t}, t') - \sum_m \int_{-\infty}^{t'} d\bar{t} (\Sigma_c)_{nm}^{\geq}(t, \bar{t}) \mathbf{A}_{ml}(\bar{t}, t'). \end{aligned} \quad (4.2.3)$$

Here we introduced the inverse of the non-interacting 2×2 matrix Bose gas propagator

$$(\mathbf{G}_0)_{nm}^{-1}(t, t') = \left[i\tau_3 \frac{\partial}{\partial t} - \epsilon_n + \mu \right] \mathbb{1} \delta_{nm} \delta(t - t') \equiv (\mathbf{G}_0)^{-1}(t; n) \cdot \delta_{nm} \delta(t - t'), \quad (4.2.4)$$

where

$$\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4.2.5)$$

and $\mathbb{1}$ is the 2×2 identity matrix. Further the non-condensate particle is defined as follows

$$\mathbf{G}'_{ml}(t, t') = -i \begin{pmatrix} \langle T_c b_n(t) b_m^\dagger(t') \rangle & \langle T_c b_n(t) b_m(t') \rangle \\ \langle T_c b_n^\dagger(t) b_m^\dagger(t') \rangle & \langle T_c b_n^\dagger(t) b_m(t') \rangle \end{pmatrix}, \quad (4.2.6)$$

with T_c now as the ordering along the Keldysh contour. The Hartree-Fock selfenergy Σ_{nm}^{HF} can be written down according to the Feynman rules extracted from the Hamiltonian (4.1.13). Its components are

$$\begin{aligned} \Sigma_{11}^{HF}(t) &= CN_0(t) + D(a_1^*(t)a_2(t) + a_2^*(t)a_1(t)) + 2B \sum_m \langle b_m^\dagger(t+)b_m(t) \rangle \\ \Sigma_{12}^{HF}(t) &= \frac{C}{2} \sum_{i=1}^2 a_i(t)a_i(t) + Da_1(t)a_2(t) + B \sum_m \langle b_m(t+)b_m(t) \rangle \\ \Sigma_{21}^{HF}(t) &= \frac{C}{2} \sum_{i=1}^2 a_i^*(t)a_i^*(t) + Da_1^*(t)a_2^*(t) + B \sum_m \langle b_m^\dagger(t+)b_m^\dagger(t) \rangle \\ \Sigma_{22}^{HF}(t) &= CN_0(t) + D(a_1^*(t)a_2(t) + a_2^*(t)a_1(t)) + 2B \sum_m \langle b_m^\dagger(t+)b_m(t) \rangle \end{aligned} \quad (4.2.7)$$

where

$$\Sigma_{nm}^{HF}(t, t') = \Sigma^{HF}(1) \cdot \delta_{nm} \delta(t - t'). \quad (4.2.8)$$

The two terms containing convolutions of the selfenergies (of higher order processes) with the non-condensate particle Green's function on the right hand side of Eq. (4.2.3) were included phenomenologically in order to be able to study thermalization effects due to collisions (in future works).

Now, following a similar analysis like in Chapter 2. We start by writing down the 11 and 12 components of Eq. (4.2.3) and from its hermitian conjugate

$$\begin{aligned} \left[i \frac{\partial}{\partial t} - \epsilon_n + \mu \right] G_{nm}^<(t, t') &= \Sigma_{11}^{HF}(t) G_{nm}^<(t, t') + \Sigma_{12}^{HF}(t) F_{nm}^{\dagger <}(t, t') + \\ &+ \sum_l \int_{-\infty}^t d\bar{t} \left[(\Gamma_{nl})_{11}(t, \bar{t}) G_{lm}^<(\bar{t}, t') + (\Gamma_{nl})_{12}(t, \bar{t}) F_{lm}^{\dagger <}(\bar{t}, t') \right] \\ &- \sum_l \int_{-\infty}^{t'} d\bar{t} \left[(\Sigma_{nl}^{<})_{11}(t, \bar{t}) (A_{lm})_{11}(\bar{t}, t') + (\Sigma_{nl}^{<})_{12}(t, \bar{t}) (A_{lm})_{21}(\bar{t}, t') \right], \end{aligned} \quad (4.2.9)$$

$$\begin{aligned}
\left[i \frac{\partial}{\partial t} - \epsilon_n + \mu \right] F_{nm}^<(t, t') &= \Sigma_{11}^{HF}(t) F_{nm}^<(t, t') + \Sigma_{12}^{HF}(t) \overline{G}_{nm}^<(t, t') \\
&+ \sum_l \int_{-\infty}^t d\bar{t} \left[(\Gamma_{nl})_{11}(t, \bar{t}) F_{lm}^<(\bar{t}, t') + (\Gamma_{nl})_{12}(t, \bar{t}) \overline{G}_{nm}^<(\bar{t}, t') \right] \\
&- \sum_l \int_{-\infty}^{t'} d\bar{t} \left[(\Sigma_{nl}^{c<})_{11}(t, \bar{t}) (A_{lm})_{12}(\bar{t}, t') + (\Sigma_{nl}^{c<})_{12}(t, \bar{t}) (A_{lm})_{22}(\bar{t}, t') \right] \quad (4.2.10)
\end{aligned}$$

and

$$\begin{aligned}
\left[-i \frac{\partial}{\partial t'} - \epsilon_m + \mu \right] G_{nm}^<(t, t') &= \Sigma_{11}^{HF}(t') G_{nm}^<(t, t') + \Sigma_{21}^{HF}(t') F_{nm}^<(t, t') + \\
&- \sum_l \int_{-\infty}^{t'} d\bar{t} \left[G_{nl}^<(t, \bar{t}) (\Gamma_{lm})_{11}(\bar{t}, t') + F_{nl}^<(t, \bar{t}) (\Gamma_{lm})_{21}(\bar{t}, t') \right] \\
&+ \sum_l \int_{-\infty}^t d\bar{t} \left[(A_{lm})_{11}(t, \bar{t}) (\Sigma_{nl}^{c<})_{11}(\bar{t}, t') + (A_{lm})_{12}(t, \bar{t}) (\Sigma_{nl}^{c<})_{21}(\bar{t}, t') \right], \quad (4.2.11)
\end{aligned}$$

$$\begin{aligned}
\left[i \frac{\partial}{\partial t'} - \epsilon_m + \mu \right] F_{nm}^<(t, t') &= \Sigma_{22}^{HF}(t') F_{nm}^<(t, t') + \Sigma_{12}^{HF}(t') G_{nm}^<(t, t') \\
&- \sum_l \int_{-\infty}^{t'} d\bar{t} \left[F_{nl}^<(t, \bar{t}) (\Gamma_{lm})_{22}(\bar{t}, t') + G_{nl}^<(t, \bar{t}) (\Gamma_{lm})_{12}(\bar{t}, t') \right] \\
&+ \sum_l \int_{-\infty}^t d\bar{t} \left[(A_{nl})_{11}(t, \bar{t}) (\Sigma_{lm}^{c<})_{12}(\bar{t}, t') + (A_{nl})_{12}(t, \bar{t}) (\Sigma_{lm}^{c<})_{22}(\bar{t}, t') \right] \quad (4.2.12)
\end{aligned}$$

where we introduced the components of the non-condensate particle Green's function as follows

$$\mathbf{G}'_{ml}(t, t') = \begin{pmatrix} G_{nm}(t, t') & F_{nm}(t, t') \\ F_{nm}^\dagger(t, t') & \overline{G}_{nm}(t, t') \end{pmatrix}. \quad (4.2.13)$$

The equations for the greater component look the same, except that $>$ appears instead of $<$ as superscript.

In the same way, as it was done in Chapter 2, we take the difference of Eq. (4.2.9) and Eq. (4.2.11)

$$\begin{aligned}
&\left[i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial t'} \right) - (\epsilon_n - \epsilon_m) - (\Sigma_{11}^{HF}(t) - \Sigma_{11}^{HF}(t')) \right] G_{nm}^<(t, t') \\
&= \Sigma_{12}^{HF}(t) F_{nm}^\dagger<(t, t') - \Sigma_{21}^{HF}(t') F_{nm}^<(t, t') + (I_{nm}^<)_{11}(t, t') \quad (4.2.14)
\end{aligned}$$

and then the sum of Eq. (4.2.10) and Eq. (4.2.12)

$$\begin{aligned} & \left[i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial t'} \right) - (\epsilon_n + \epsilon_m) + 2\mu - (\Sigma_{11}^{HF}(t) + \Sigma_{22}^{HF}(t')) \right] F_{nm}^<(t, t') \\ &= \Sigma_{12}^{HF}(t) \bar{G}_{nm}^<(t, t') + \Sigma_{12}^{HF}(t') G_{nm}^<(t, t') + (I_{nm}^<)_{12}(t, t'), \end{aligned} \quad (4.2.15)$$

where we introduced $(I_{nm}^<)_{11}$ and $(I_{nm}^<)_{12}$ like in Chapter 2.²

Now we express the last two equations for the components of the non-condensate particle Green's function in terms of the center of mass time variable

$$T = \frac{t + t'}{2}, \quad (4.2.16)$$

and the relative time coordinate

$$\tau = t - t'. \quad (4.2.17)$$

Further we assume, that the Hartree-Fock self-energies depend only on small values of τ and therefore we can expand them in powers of τ , keeping only the linear terms. Thus we obtain

$$\begin{aligned} & \left[i \frac{\partial}{\partial T} - (\epsilon_n - \epsilon_m) - \tau \frac{\partial}{\partial T} \Sigma_{11}^{HF}(T) \right] G_{nm}^<(\tau, T) \\ &= \Sigma_{12}^{HF}(T) F_{nm}^{\dagger <}(\tau, T) - \Sigma_{21}^{HF}(T) F_{nm}^<(\tau, T) \\ &+ \frac{1}{2} \left[F_{nm}^{\dagger <}(\tau, T) \tau \frac{\partial}{\partial T} \Sigma_{12}^{HF}(T) + F_{nm}^<(\tau, T) \tau \frac{\partial}{\partial T} \Sigma_{21}^{HF}(T) \right] \\ &+ (I_{nm}^<)_{11}(\tau, T) \end{aligned} \quad (4.2.18)$$

and

$$\begin{aligned} & \left[i \frac{\partial}{\partial T} - (\epsilon_n + \epsilon_m) + 2\mu - 2\Sigma_{11}^{HF}(T) \right] F_{nm}^<(\tau, T) \\ &= \Sigma_{12}^{HF}(T) \bar{G}_{nm}^<(\tau, T) + \Sigma_{12}^{HF}(T) G_{nm}^<(\tau, T) \\ &+ \frac{1}{2} \left[\bar{G}_{nm}^<(\tau, T) \tau \frac{\partial}{\partial T} \Sigma_{12}^{HF}(T) - G_{nm}^<(\tau, T) \tau \frac{\partial}{\partial T} \Sigma_{12}^{HF}(T) \right] \\ &+ (I_{nm}^<)_{12}(\tau, T). \end{aligned} \quad (4.2.19)$$

The last step in the derivation of the kinetic equations for the non-condensate particle Green's function is to do the Fourier transform with respect to τ of Eq.

²A more detailed discussion can be found in Appendix B.

(4.2.18) and Eq. (4.2.19). We get now

$$\begin{aligned}
& \left[\frac{\partial}{\partial T} + i(\epsilon_n - \epsilon_m) + \frac{\partial}{\partial T} \Sigma_{11}^{HF}(T) \frac{\partial}{\partial \omega} \right] G_{nm}^<(\omega, T) \\
&= -i \left[\Sigma_{12}^{HF}(T) F_{nm}^{\dagger <}(\omega, T) - \Sigma_{21}^{HF}(T) F_{nm}^<(\omega, T) \right] \\
&\quad - \frac{1}{2} \left[\frac{\partial \Sigma_{12}^{HF}(T)}{\partial T} \frac{\partial}{\partial \omega} F_{nm}^{\dagger <}(\omega, T) + \frac{\partial \Sigma_{21}^{HF}(T)}{\partial T} \frac{\partial}{\partial \omega} F_{nm}^<(\omega, T) \right] \\
&\quad - i(I_{nm}^<)_{11}(\omega, T), \tag{4.2.20}
\end{aligned}$$

$$\begin{aligned}
& \left[\frac{\partial}{\partial T} + i(\epsilon_n + \epsilon_m) - i2\mu + i2\Sigma_{11}^{HF}(T) \right] F_{nm}^<(\omega, T) \\
&= -i\Sigma_{12}^{HF}(T) \left[\overline{G}_{nm}^<(\omega, T) + G_{nm}^<(\omega, T) \right] \\
&\quad - \frac{1}{2} \frac{\partial \Sigma_{12}^{HF}(T)}{\partial T} \frac{\partial}{\partial \omega} \left[\overline{G}_{nm}^<(\omega, T) - G_{nm}^<(\omega, T) \right] \\
&\quad - i(I_{nm}^<)_{12}(\omega, T). \tag{4.2.21}
\end{aligned}$$

The Fourier components of the lesser Green's function appearing in the equations above are given by

$$B_{nm}^<(\tau, T) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega\tau} B_{nm}^<(\omega, T), \quad \text{for } B \in \{G, F, F^\dagger, \overline{G}\}. \tag{4.2.22}$$

Eq. (4.2.20) and Eq. (4.2.21) are the kinetic equations for the non-condensate Green's function and anomalous Green's function. Now in order to have a system of self-consistent coupled equations, we have to derive the equations of motion for the two macroscopic wave functions. For this we make use of the generalized Gross-Pitaevskii equation (2.2.42). We then decompose the order parameter in two modes

$$\Psi_0(\vec{x}, T) = \phi_1(\vec{R})a_1(T) + \phi_2(\vec{R})a_2(T) \tag{4.2.23}$$

and replace the operators Ψ' and Ψ'^\dagger appearing in the non-condensate particle density and anomalous density by the mode expansion displayed in Eq. (4.1.2), i.e.

$$\Psi'(\vec{R}, T) = \sum_n \varphi_n(\vec{R})b_n(T). \tag{4.2.24}$$

Notice, that the arguments appearing in the field operator are the macroscopic coordinates. This is valid, since the dependences in the the field operators appearing in the average, that corresponds to the density of the non-condensate particles, are

the same. Making the replacements (4.2.23) and (4.2.24) in Eq. (2.2.42), then multiplying by $\phi_1(\vec{R})$ and integrating over \vec{R} , we obtain

$$\begin{aligned}
i\frac{\partial}{\partial T}a_1(T) &= \left(E - \mu + T_0N_1(T) + iC \sum_n iG_{nn}^<(\tau = 0, T) \right) a_1(T) \\
&- \left(\frac{J}{2} - D \sum_n iG_{nn}^<(\tau = 0, T) \right) a_2(T) + \left(\frac{C}{2}a_1^*(T) + \frac{D}{2}a_2^* \right) \sum_n iF_{nn}^<(\tau = 0, T) \\
&+ \left(\int d^3R (\phi_1(\vec{R}))^2 \int_{-\infty}^T d2 [S_{11}^> - S_{11}^<] (\vec{R} - \vec{r}_2, T - t_2) \right) a_1(T) + \\
&+ \left(\int d^3R \phi_1(\vec{R})\phi_2(\vec{R}) \int_{-\infty}^T d2 [S_{11}^> - S_{11}^<] (\vec{R} - \vec{r}_2, T - t_2) \right) a_2(T) + \\
&+ \left(\int d^3R (\phi_1(\vec{R}))^2 \int_{-\infty}^T d2 [S_{12}^> - S_{12}^<] (\vec{R} - \vec{r}_2, T - t_2) \right) a_1^*(T) + \\
&+ \left(\int d^3R \phi_1(\vec{R})\phi_2(\vec{R}) \int_{-\infty}^T d2 [S_{12}^> - S_{12}^<] (\vec{R} - \vec{r}_2, T - t_2) \right) a_2^*(T). \quad (4.2.25)
\end{aligned}$$

Similarly, after multiplying by $\phi_2(\vec{R})$ and integrating over \vec{R} , we obtain

$$\begin{aligned}
i\frac{\partial}{\partial T}a_2(T) &= \left(E - \mu + T_0N_2(T) + iC \sum_n iG_{nn}^<(\tau = 0, T) \right) a_2(T) \\
&- \left(\frac{J}{2} - D \sum_n iG_{nn}^<(\tau = 0, T) \right) a_1(T) + \left(\frac{C}{2}a_2^*(T) + \frac{D}{2}a_1^* \right) \sum_n iF_{nn}^<(\tau = 0, T) \\
&+ \left(\int d^3R \phi_2(\vec{R}) \int_{-\infty}^T d2 [S_{11}^> - S_{11}^<] (\vec{R} - \vec{r}_2, T - t_2) \right) a_2(T) + \\
&+ \left(\int d^3R \phi_1(\vec{R})\phi_2(\vec{R}) \int_{-\infty}^T d2 [S_{11}^> - S_{11}^<] (\vec{R} - \vec{r}_2, T - t_2) \right) a_1(T) + \\
&+ \left(\int d^3R \phi_2(\vec{R}) \int_{-\infty}^T d2 [S_{12}^> - S_{12}^<] (\vec{R} - \vec{r}_2, T - t_2) \right) a_2^*(T) + \\
&+ \left(\int d^3R \phi_1(\vec{R})\phi_2(\vec{R}) \int_{-\infty}^T d2 [S_{12}^> - S_{12}^<] (\vec{R} - \vec{r}_2, T - t_2) \right) a_1^*(T). \quad (4.2.26)
\end{aligned}$$

where we used the approximations explained above.

The Eqs. (4.2.20) and (4.2.21) for the Green's function of the non-condensate particles and the Eqs. (4.2.25) and Eq. (4.2.26) for the two macroscopic wave functions form a system of coupled integro-differential equation, that describe the non-equilibrium features of two dilute Bose gas confined in a double well trap. The integrals on the right hand side of all these equations suggest thermalization of the excited particles due to collisions and the Hartree-Fock self-energies display density fluctuations.

4.2.1 The Collisionless Regime

Above we presented the model we want to consider. We used a combination of Keldysh technique for systems out of equilibrium and field theoretical methods for bosons at zero temperature for the derivation of the system of coupled differential equations describing the model under consideration. This system of equations consists of the kinetic equations for the Green's function and the anomalous Green's function of the non-condensate particles, and the generalized Gross-Pitaevskii equations for the two condensate modes.

Now we are going to compute the equations for the collisionless regime. And for this we follow the analysis done in section 2.2.1. We start by neglecting the collision term and by putting the relative time coordinate τ in Eq. (4.2.18) and Eq. (4.2.21) equal to zero. Thus we obtain

$$\left[i \frac{\partial}{\partial T} - (\epsilon_n - \epsilon_m) \right] G_{nm}^<(\tau = 0, T) = \Sigma_{12}^{HF}(T) F_{nm}^{\dagger <}(\tau = 0, T) - \Sigma_{21}^{HF}(T) F_{nm}^<(\tau = 0, T)$$

and

$$\begin{aligned} & \left[i \frac{\partial}{\partial T} - (\epsilon_n + \epsilon_m) + 2\mu - 2\Sigma_{11}^{HF}(T) \right] F_{nm}^<(\tau = 0, T) \\ & = \Sigma_{12}^{HF}(T) \overline{G}_{nm}^<(\tau = 0, T) + \Sigma_{12}^{HF}(T) G_{nm}^<(\tau = 0, T). \end{aligned} \quad (4.2.27)$$

Now we put $n = m$ and define the distribution functions

$$f_1(T, n) = iG_{nn}^<(\tau = 0, T) \quad (4.2.28)$$

$$f_2(T, n) = iF_{nn}^<(\tau = 0, T), \quad (4.2.29)$$

and rewrite the kinetic equations:

$$\frac{\partial}{\partial T} f_1(T, n) = -i \left[\Sigma_{12}^{HF}(T) f_2^*(T, n) - \Sigma_{21}^{HF}(T) f_2(T, n) \right]$$

and

$$\left[\frac{\partial}{\partial T} + i2(\epsilon_n - \mu + \Sigma_{11}^{HF}(T)) \right] f_2(T, n) = -i \Sigma_{12}^{HF}(T) (1 + 2f_1(T, n)), \quad (4.2.30)$$

which are now coupled to

$$\begin{aligned}
i \frac{\partial}{\partial T} a_1(T) &= \left(E - \mu + T_0 N_1(T) + C \sum_n f_1(T, n) \right) a_1(T) \\
&\quad - \left(\frac{J}{2} - D \sum_n f_1(T, n) \right) a_2(T) + \left(\frac{C}{2} a_1^*(T) + \frac{D}{2} a_2^* \right) \sum_n f_2(T, n) \quad (4.2.31)
\end{aligned}$$

and

$$\begin{aligned}
i \frac{\partial}{\partial T} a_2(T) &= \left(E - \mu + T_0 N_2(T) + C \sum_n f_1(T, n) \right) a_2(T) \\
&\quad - \left(\frac{J}{2} - D \sum_n f_1(T, n) \right) a_1(T) + \left(\frac{C}{2} a_2^*(T) + \frac{D}{2} a_1^* \right) \sum_n f_2(T, n). \quad (4.2.32)
\end{aligned}$$

In the next chapter we will present the numerical solutions of this system of equations.

Chapter 5

Results

In the preceding chapter we performed the microscopic derivation of the Hamiltonian describing two Bose gases confined in a double well trap at zero temperature. We also derived the corresponding kinetic equations for the non-condensate particles and its associated equations for the two macroscopic condensate wave functions. At the end of the chapter, in section 4.2.1 we considered the collisionless regime, which is described within the Hartree-Fock approximation. In this chapter we want to present the numerical results of these equations and discuss them.

5.1 The Boundary conditions

We consider first a system consisting of two independent Bose gases confined in a double well potential at zero temperature. The system is assumed to be in equilibrium, so that all particles of both gases are occupying the lowest state of energy. Due to interactions, each of the gases can present particles occupying higher states of energy. The analysis presented in [44] showed that at zero temperature this quantum depletion can be $\lesssim 0.5\%$. This depletion can appear for each condensate amplitude in our model, and is displayed by a non-zero intrawell coupling C between the condensate particles and the non-condensate particles. At a time $t = t_0$ we suddenly change the shape of the trapping potential, lowering the barrier separating the two condensates. The experimental realization of such a set up is not difficult [15, 13]. We have explained before, that one of the possible realizations of a double well trapping potential is the overlap of a three dimensional harmonic confinement, generated by the magneto-optical trap, and a periodic one-dimensional optical lattice with large lattice spacing. In order to change the height and/or width of the barrier one has to modulate the amplitude and/or the lattice spacing of the optical lattice. This would change the value of the Josephson coupling J , which is related to the probability of tunneling from one well to the other one for one particle. Moreover, upon suddenly lowering the barrier between the condensates, these start mixing, and Bogoliubov quasiparticles above the condensates are cre-

ated, absorbing part of the system's entropy.¹ In the case, that the barrier was not lowered sufficiently, so that the particle tunneling through the barrier can set in, we will see that the mixing of the gases is possible for a large coupling D describing an *auxiliary bypass* over the single-particle excitations.

Before we start the discussion of our results, we have to point out that we express all the energies in terms of the Josephson coupling J . Therefore we can write for the Bogoliubov-Hartree-Fock coupled equations (4.2.30) - (4.2.32)

$$\frac{\partial}{\partial T} f_1(T, n) = -i [\Sigma_{12}^{HF}(T) f_2^*(T, n) - \Sigma_{21}^{HF}(T) f_2(T, n)] \quad (5.1.1)$$

and

$$\left[\frac{\partial}{\partial T} + i2 (\tilde{\epsilon}_n - \tilde{\mu} + \Sigma_{11}^{HF}(T)) \right] f_2(T, n) = -i \Sigma_{12}^{HF}(T) (1 + 2f_1(T, n)), \quad (5.1.2)$$

with

$$\begin{aligned} \Sigma_{11}^{HF}(T) &= \tilde{C} N_0(T) + \tilde{D} (a_1^*(T) a_2(T) + a_2^*(T) a_1(T)) + 2\tilde{B} \sum_m f_1(T, m) \\ \Sigma_{12}^{HF}(T) &= \frac{\tilde{C}}{2} \sum_{i=1}^2 a_i(T) a_i(T) + \tilde{D} a_1(T) a_2(T) + \tilde{B} \sum_m f_2(T, m) \\ \Sigma_{21}^{HF}(T) &= \frac{\tilde{C}}{2} \sum_{i=1}^2 a_i^*(T) a_i^*(T) + \tilde{D} a_1^*(T) a_2^*(T) + \tilde{B} \sum_m f_2^*(T, m) \\ \Sigma_{22}^{HF}(T) &= \Sigma_{11}^{HF}(T), \end{aligned} \quad (5.1.3)$$

and for the associated equations for the macroscopic wave functions

$$\begin{aligned} i \frac{\partial}{\partial T} a_1(T) &= \left(\tilde{E} - \tilde{\mu} + \tilde{T}_0 N_1(T) + \tilde{C} \sum_n f_1(T, n) \right) a_1(T) \\ &\quad - \left(1 - \tilde{D} \sum_n f_1(T, n) \right) a_2(T) + \left(\frac{\tilde{C}}{2} a_1^*(T) + \frac{\tilde{D}}{2} a_2^* \right) \sum_n f_2(T, n) \end{aligned} \quad (5.1.4)$$

and

$$\begin{aligned} i \frac{\partial}{\partial T} a_2(T) &= \left(\tilde{E} - \tilde{\mu} + \tilde{T}_0 N_2(T) + \tilde{C} \sum_n f_1(T, n) \right) a_2(T) \\ &\quad - \left(1 - \tilde{D} \sum_n f_1(T, n) \right) a_1(T) + \left(\frac{\tilde{C}}{2} a_2^*(T) + \frac{\tilde{D}}{2} a_1^* \right) \sum_n f_2(T, n), \end{aligned} \quad (5.1.5)$$

¹Experimentally two independently prepared Bose gases ($J = 0$) have a definite particle number and are therefore described by a coherent superposition of all possible phases between 0 and 2π . If one permits at some certain point $t = t_0$ the exchange of particles between the condensates, decoherence sets in and the relative phase difference between both condensates takes random values.

where

$$\tilde{A} = \frac{A}{J/2} \quad \text{for } A \in \{\epsilon_n, \mu, E, T_0, C, D, B\}, \quad (5.1.6)$$

and the time was rescaled as follows

$$\frac{J}{2}T \mapsto T. \quad (5.1.7)$$

For the discussion we now introduce the particle imbalance between the two condensates normalized to the total particle number

$$z = \frac{N_1 - N_2}{N_1 + N_2 + \sum_n f_1(T, n)} = \frac{N_1 - N_2}{N}. \quad (5.1.8)$$

We keep the definition given in chapter 3 for the relative phase difference. Further, without loss of generality, we can choose the energy \tilde{E} corresponding to the lowest states to be equal to the chemical potential $\tilde{\mu}$. In the following we will introduce the difference

$$\tilde{\omega}_n = \tilde{\epsilon}_n - \tilde{\mu} \quad (5.1.9)$$

as a new parameter. In order to make some estimations for $\tilde{\omega}_n$ we recall that in Section 4.1 it was assumed for the derivation of the model, that the spectrum of the non-condensate particles coincides with the spectrum of the harmonic component of the trapping potential². We can therefore write

$$\tilde{\omega}_n = \frac{\omega_{\text{ho}}}{J/2}n, \quad (5.1.10)$$

where ω_{ho} is the oscillator frequency, and experimentally $\omega_{\text{ho}} \sim 100$ Hz [10]. The experimental value for $J \sim 10$ Hz [27]. With these two values we can make the next estimation:

$$\frac{\omega_{\text{ho}}}{J/2} \sim 10. \quad (5.1.11)$$

In the following we want to discuss the results for initial non-condensate particle number equal to zero and to a finite number, separately. For both cases we present results for large and small interatomic interactions between the condensate particles. We will see, that for small coupling C and D with the non-condensate particles, the dynamics of the system approaches the mean field regime presented in chapter 3. For simplicity, an initial macroscopic phase difference is chosen to be equal to zero.

²Surely, there are intrawell excited levels corresponding to the condensate amplitude oscillations lying lower than the barrier separating the condensates, but we expect that the dynamics of mixing are mainly displayed by the levels above the barrier separating the gases.

5.1.1 Quantum Depletions in the Initial Equilibrium State

We consider two Bose gases at zero temperature with a very large number of particles. Due to the intrawell interaction we observe quantum depletions of condensate amplitudes, which manifest themselves as a finite number of non-condensate particles ($N_1 = 1 \cdot 10^5$, $N_2 = 4 \cdot 10^5$ and $\sum_n f_1(T = 0, n) = 2500$, $\sum_n f_2(T = 0, n) = 2500.5$)³. At time $t = t_0$ we lower the barrier and the intrawell non-condensate particles coupling C increases. The auxiliary bypass D and the Hartree-Fock coupling B become finite, where B is many orders of magnitude smaller than the other parameters and the dynamics of the system do not strongly depend on it. Hence we fix the value of $B = 1 \cdot 10^{-7}$.

In the mean field approximation the dynamics of the coherent particle tunneling between two condensates is described in terms of the interwell population imbalance and the relative phase difference. In order to be able to compare our results with the results obtained in different studies done in the past [27, 31, 30], we also show the dynamics of the system in terms of the population imbalance and relative phase difference introduced in Eq. (5.1.8) and (3.1.13), respectively.

Now we will consider the dynamics of the system for $T_0 = 5 \cdot 10^{-5}$ and $T_0 = 1 \cdot 10^{-5}$ separately:

- First we consider the dynamics of a system with large interatomic interactions between the condensate particles $T_0 = 5 \cdot 10^{-5}$ for different coupling with non-condensate particles C and D . Setting the auxiliary bypass and the intrawell coupling to the noncondensate particles $C = D = 10^{-5}$ to same order of magnitude as T_0 , we observe weak deviations from the mean field regime (Page 66). The particle imbalance as function of the rescaled time presents beats caused by the amplitude oscillations of both condensates. Although the system for this parameter combination is selftrapped, because the particle imbalance is oscillation around a non-zero value.

Keeping the values for $T_0 = 5 \cdot 10^{-5}$ and $C = 10^{-5}$ we increase the auxiliary bypass $D = 1.5 \cdot 10^{-5}$ (Page 67). We observe that the system presents very strong deviations from the mean field studies. The amplitude oscillations of both condensates are at the begin larger than for the combination of parameters considered before, but even though relatively small. At some time the

³Using the "quasiparticle approximation" for the initial equilibrium state, we find that

$$f_1(T, n) = -iv_n^2 \quad \text{and} \quad f_2(T, n) = -iv_n u_n,$$

where u_n and v_n are the solutions of the Bogoliubov de Gennes equation (1.3.4) and satisfy

$$u_n^2 - v_n^2 = 1.$$

In our analysis we imply that the equilibrium problem has already been solved and assume a number of non-condensate particles $\lesssim 0.5\%$ of the total particle number. Having f_1 the estimation of f_2 is straightforward.

number of particles occupying the higher states of energy increases abruptly. This causes a stronger deviation from the mean field results, where the system undergoes a transition from a selftrapped to a non-selftrapped system, making the mixing of condensates possible. For an even stronger auxiliary bypass $D = 1 \cdot 10^{-4}$ the system undergoes the transition earlier (Page 68).

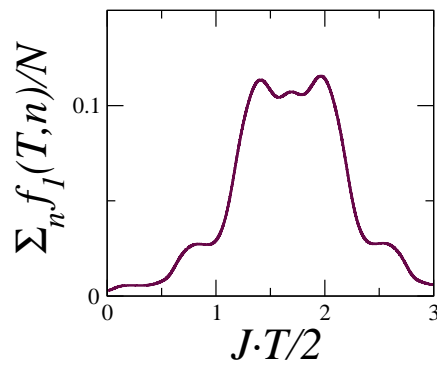
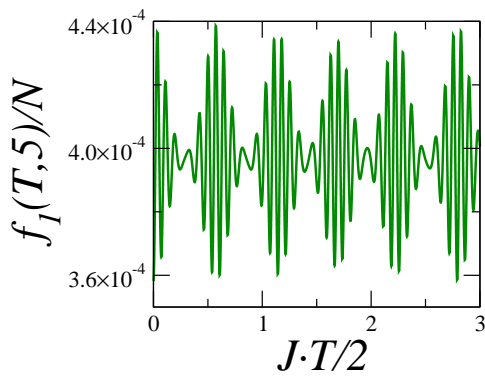
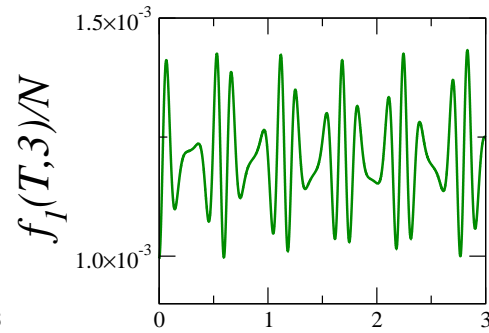
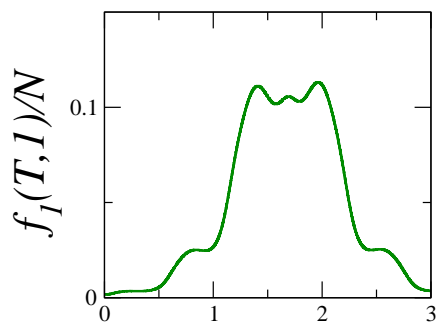
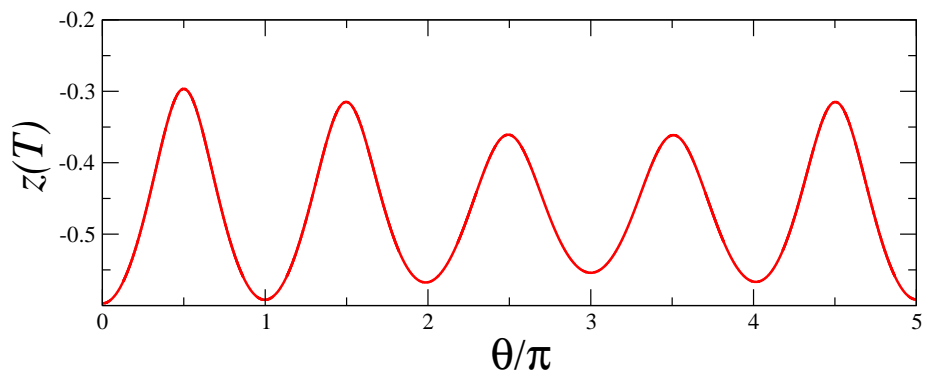
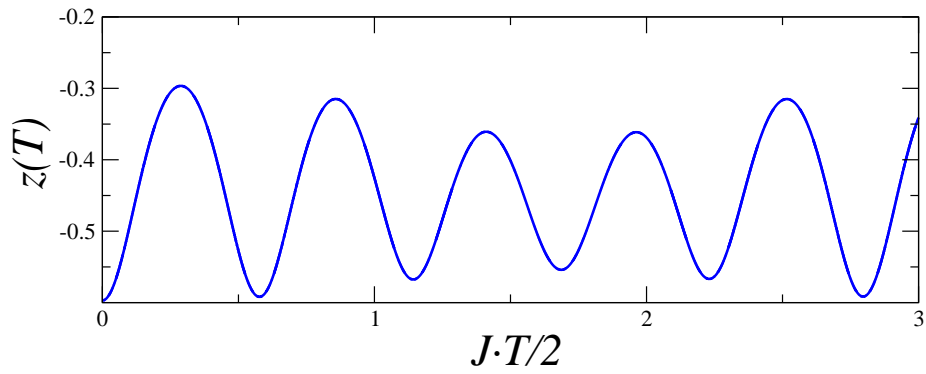
- In the same way we consider a system with a small intrawell interatomic interactions between the condensate particles $T_0 = 1 \cdot 10^{-5}$, and we set the couplings $C = 1 \cdot 10^{-5}$ and $D = 5 \cdot 10^{-5}$. In this case the system behaves as in the mean field regime (Page 69). We observe coherent particle tunneling through the barrier. The amplitude oscillations causes small deviations from the mean field behavior, that are not visible at the length scale in consideration.

Increasing the value for the auxiliary bypass $D = 7 \cdot 10^{-5}$ we find that number of non-condensate particles grows abruptly at some certian time. This causes that relative phase stops oscillating around 0 and starts oscillating around π , with a varying amplitude and a new frequency (Page 70). Making the auxiliary bypass even stronger $D = 1 \cdot 10^{-4}$, the increment in the number of non-condensate particles happens earlier and the system does not display the *glasses-like* phase space portrait anymore (Page 71).

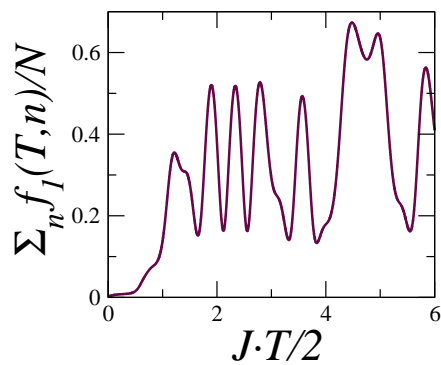
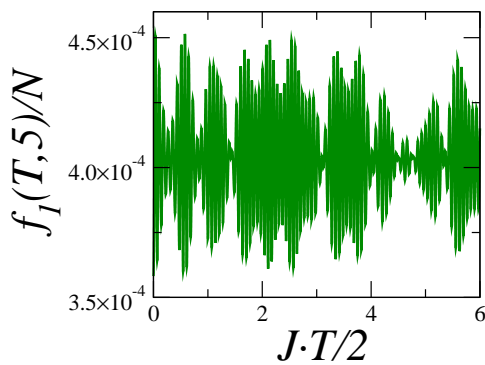
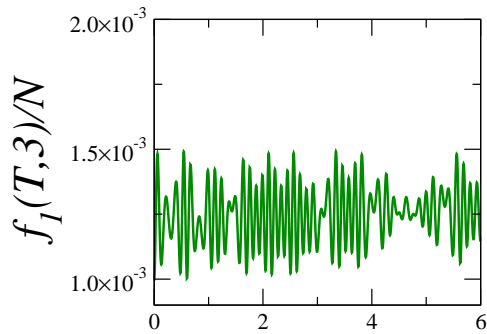
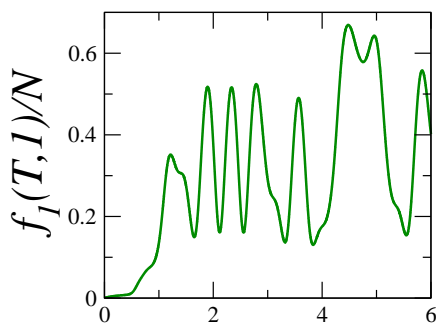
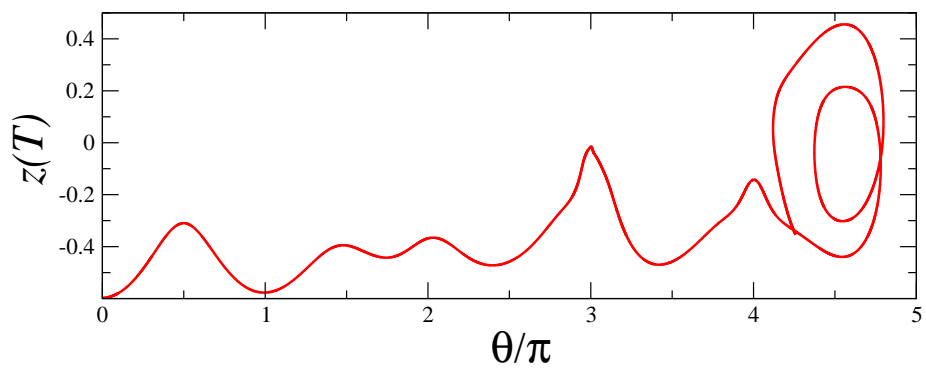
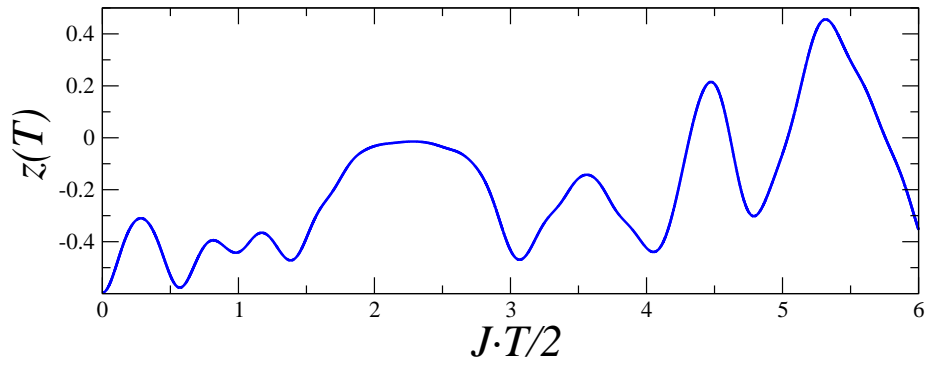
5.1.2 Initial Equilibrium State without Quantum Depletions

We mention before that at zero temperature the quantum depletion is about 0.5% (see [44]). This is a negligibly small number. We can therefore assume that at zero temperature all particles are occupying the lowest state of energy (in our system, the two condensates). Studying the system without initial non-condensate particles we find the same behavior for the different parameter sets, such as the one presented by the system with initial amplitude oscillations. Hence, we only display two different parameter combinations in order to show the system is undergoing the transition from the mean field regime to the regime where it is dominated by the amplitude oscillations (Pages 72 and 73).

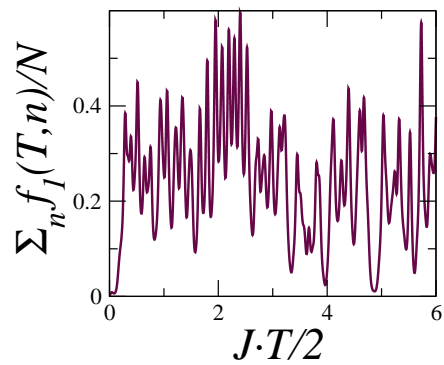
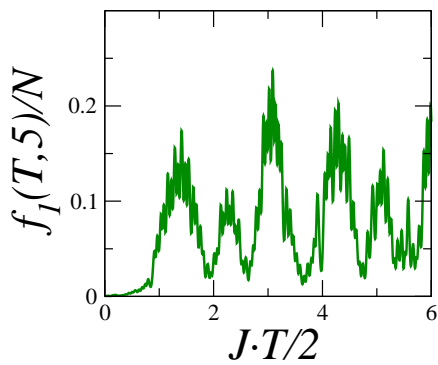
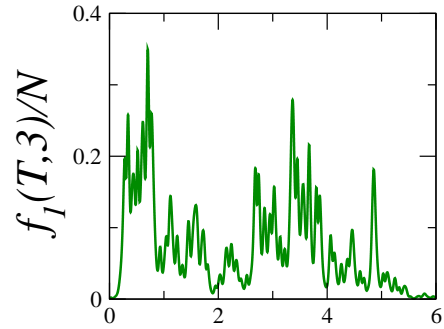
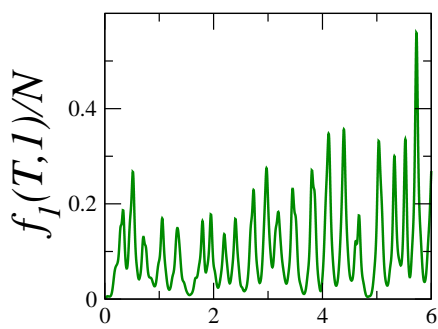
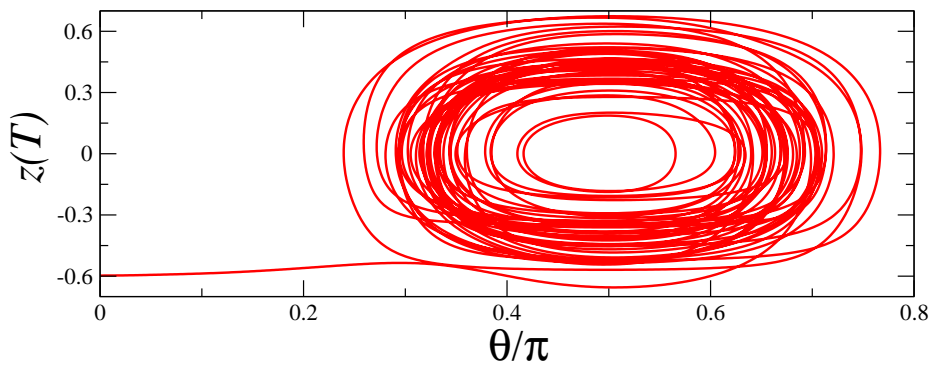
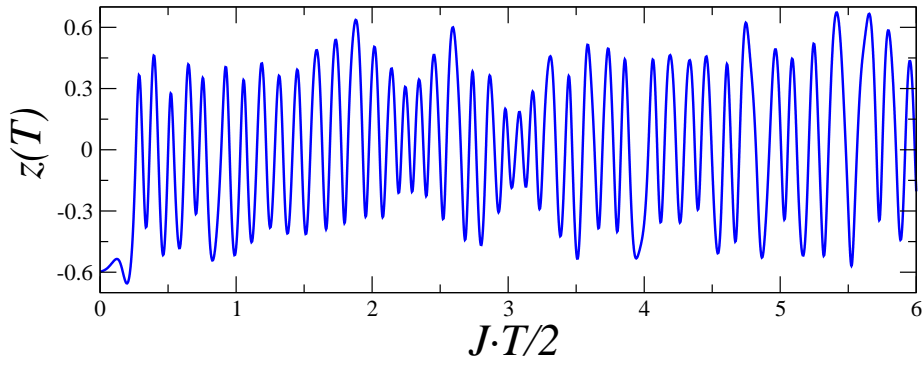
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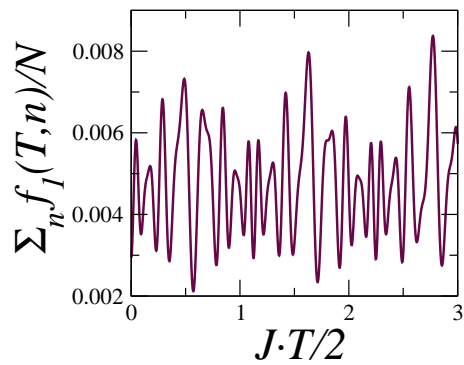
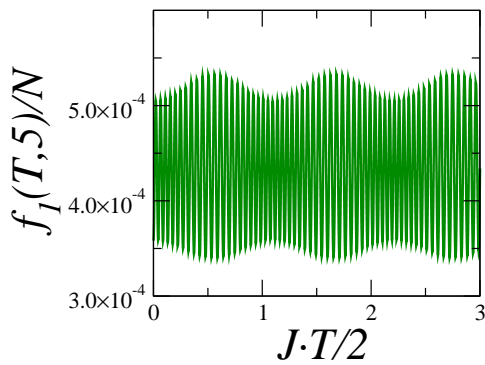
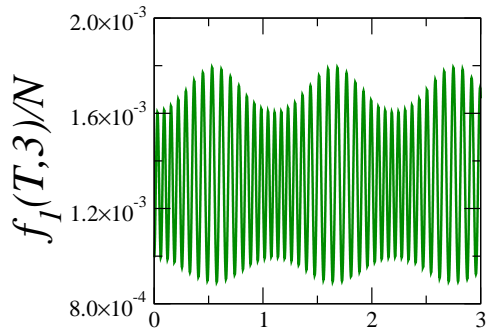
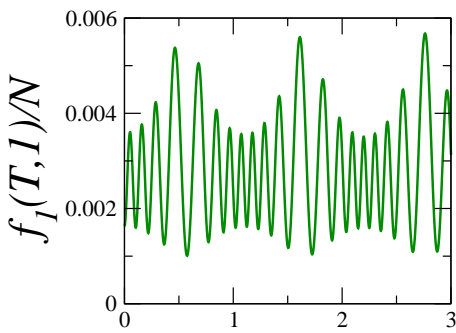
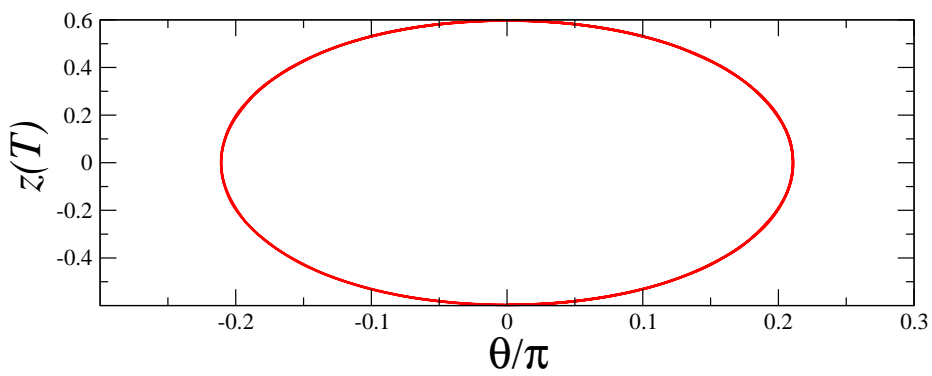
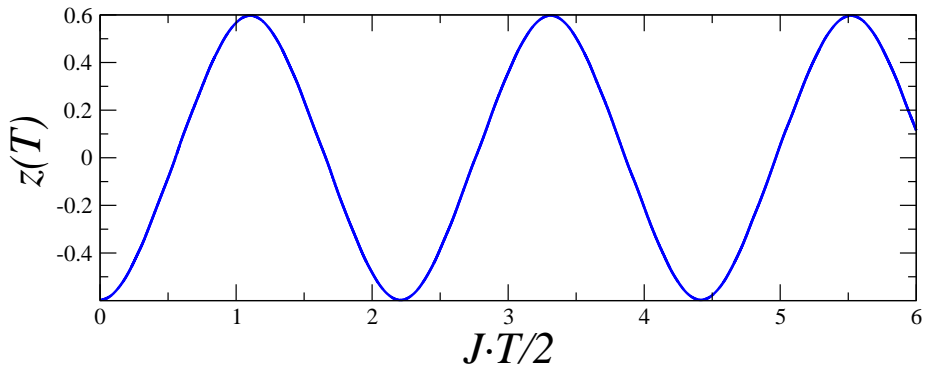
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 $C = 1 \cdot 10^{-5}$, $D = 1.5 \cdot 10^{-5}$, $B = 1 \cdot 10^{-7}$



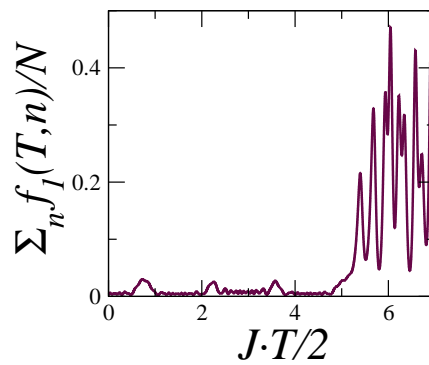
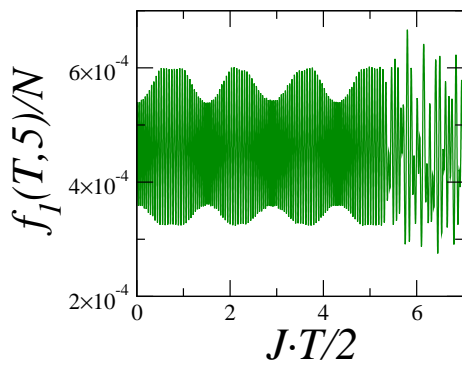
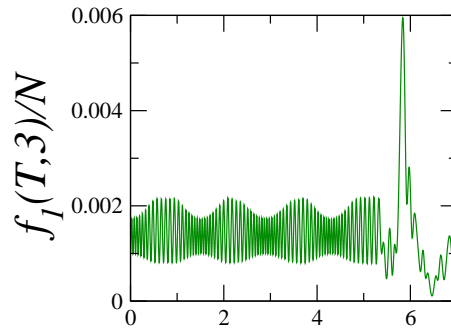
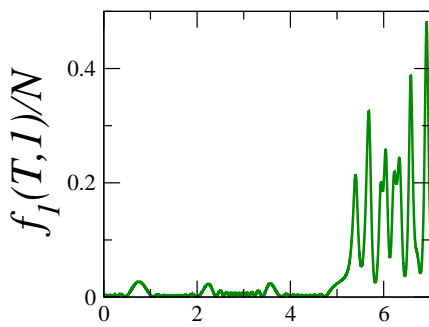
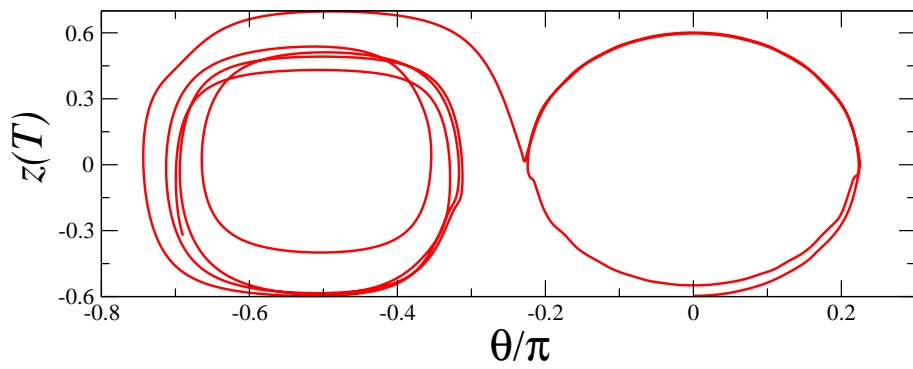
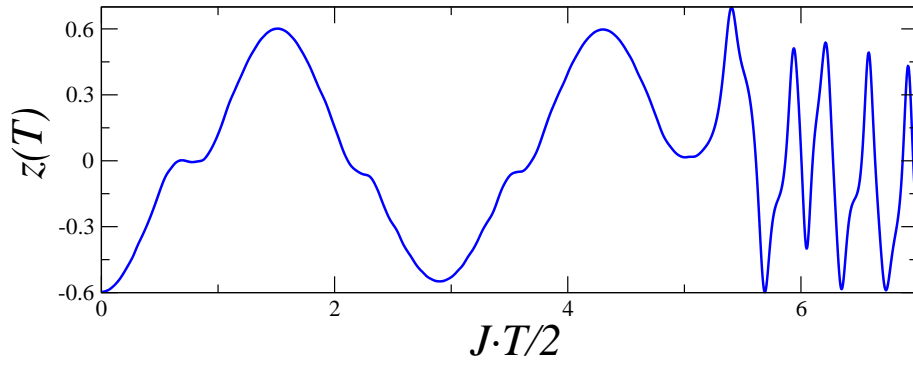
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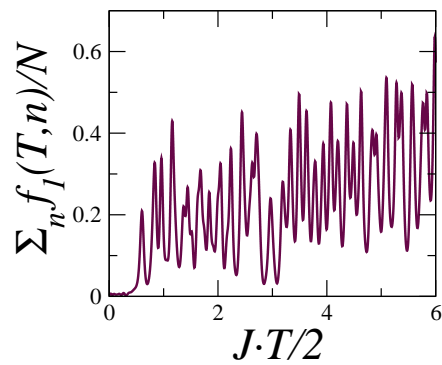
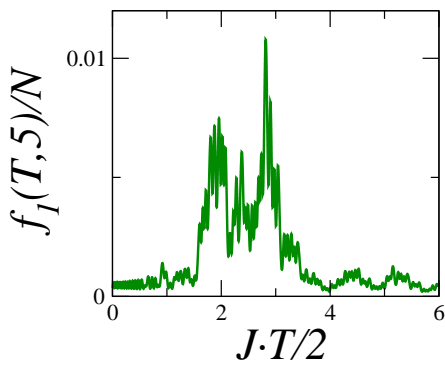
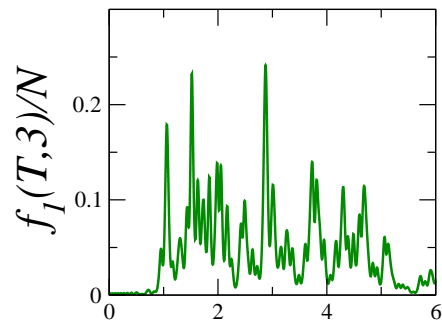
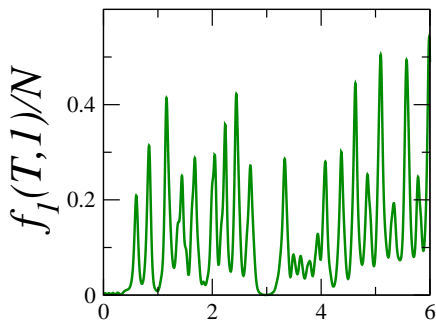
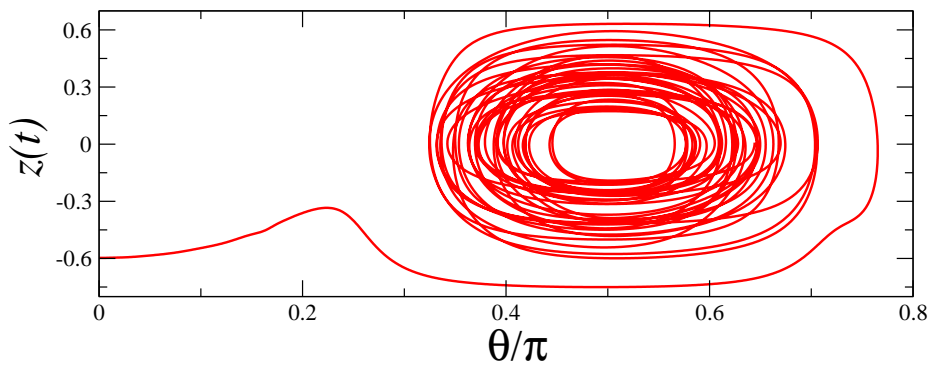
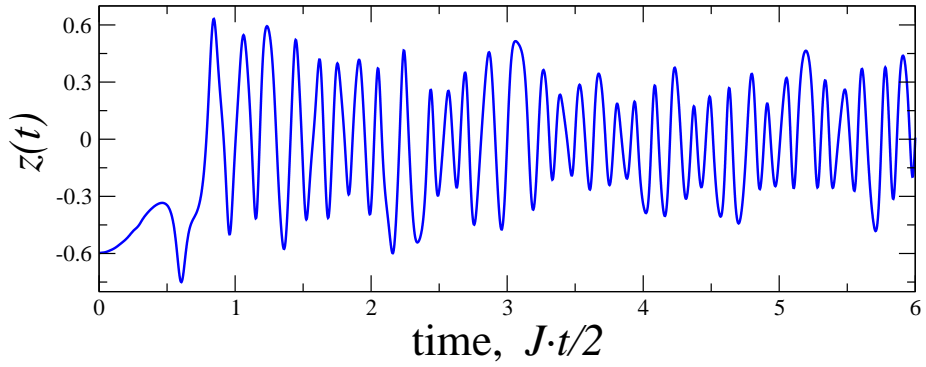
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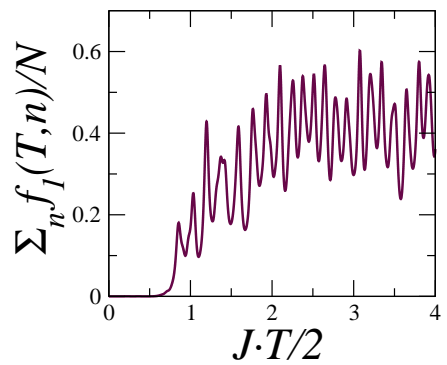
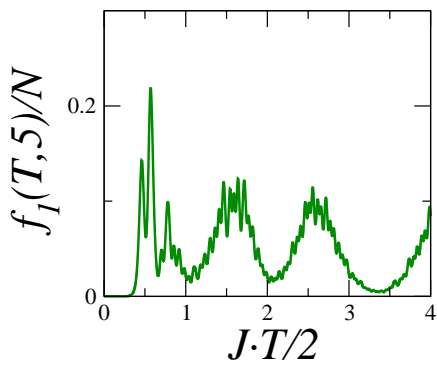
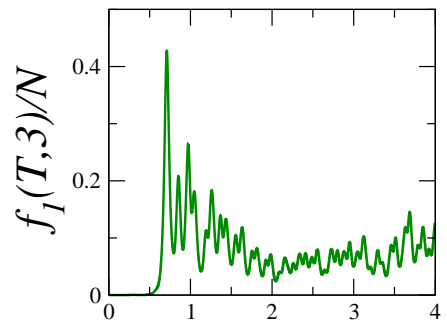
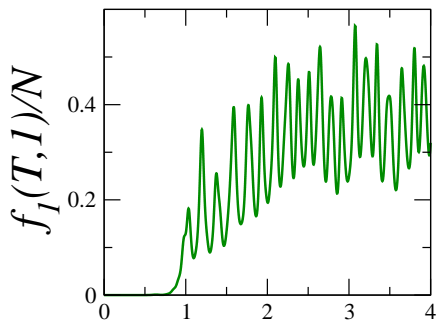
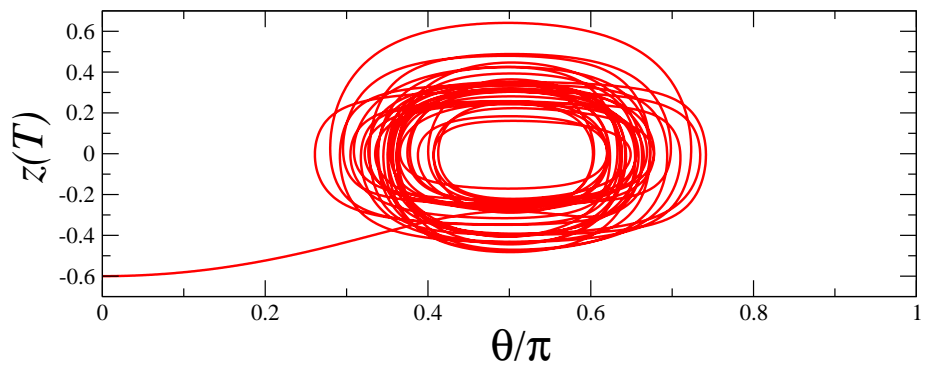
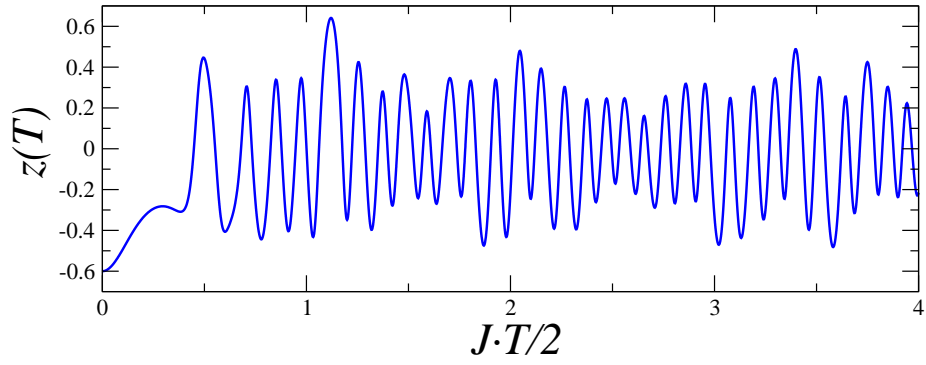
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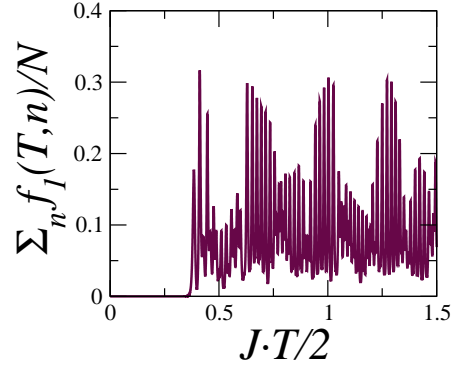
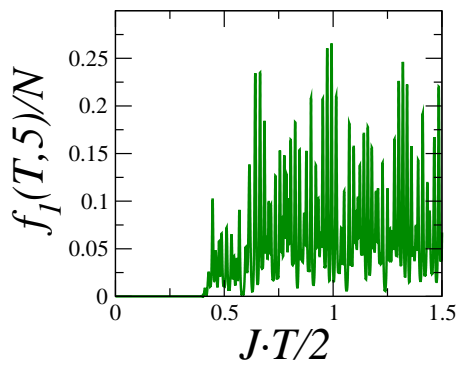
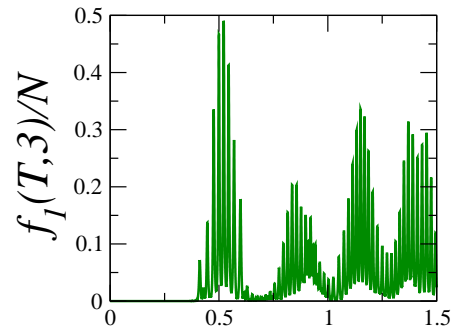
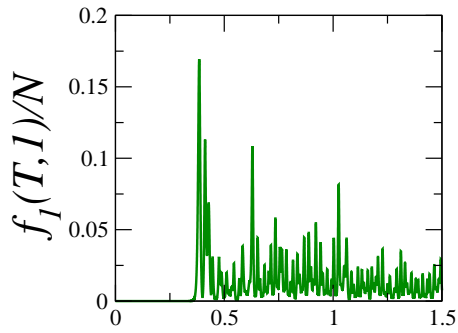
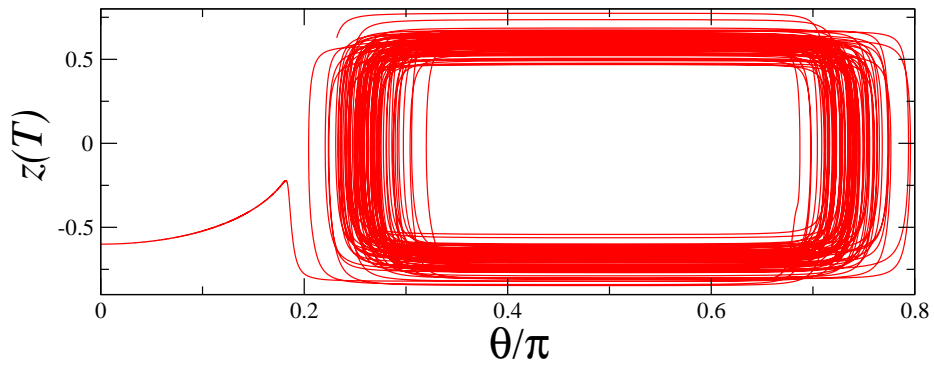
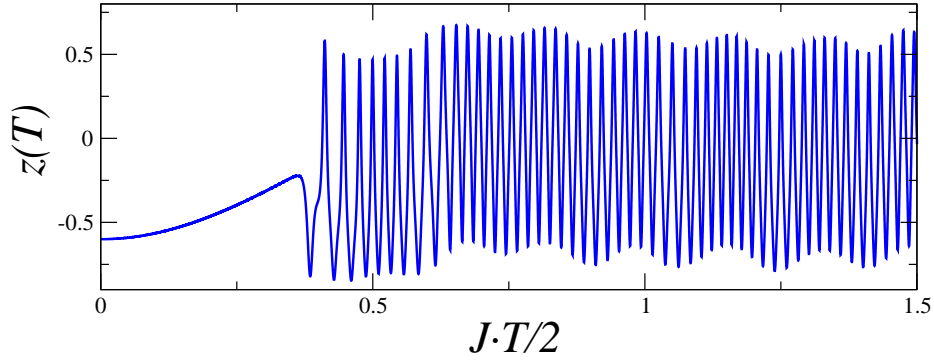
Parameters: $\sum_n f_1(T=0, n) = 2500$, $T_0 = 1 \cdot 10^{-5}$,
 $C = 1 \cdot 10^{-5}$, $D = 1 \cdot 10^{-4}$, $B = 1 \cdot 10^{-7}$



Parameters: $\sum_n f_1(T=0, n) = 0$, $T_0 = 5 \cdot 10^{-5}$,
 $C = 1 \cdot 10^{-5}$, $D = 1 \cdot 10^{-4}$, $B = 1 \cdot 10^{-6}$



Parameters: $\sum_n f_1(T=0, n) = 0$, $T_0 = 1 \cdot 10^{-5}$,
 $C = 1 \cdot 10^{-5}$, $D = 1 \cdot 10^{-3}$, $B = 1 \cdot 10^{-6}$



Conclusions

In this work we have presented a microscopic theory for the dynamics of a trapped Bose gas in the presence of a condensate fraction (see Chapters 1 and 2). Starting from the mean field approximation we derived the Gross-Pitaevskii equation. After including the quantum fluctuations we were able to present the generalized Gross-Pitaevskii equation of motion for the macroscopic condensate wave function, which is coupled to the kinetic equations for the Green's and anomalous Green's functions. The resulting equations can be applied to study a huge spectrum of problems associated with the non-equilibrium dynamics in a trapped Bose gas, such as in the present work.

We have considered a system consisting of two independently prepared trapped Bose gases at zero temperature. At time $t = t_0$ we lowered abruptly the barrier between the condensates. In this way we permitted the mixing of the condensates, and simultaneously, the appearance of single-particle excitations, which absorb part of the system's entropy. We have made use of the techniques presented in Chapter 2 for the derivation of the kinetic equations and the associated generalized Gross-Pitaevskii equation to study the problem. In Chapter 4 we derived the Hamiltonian governing of the dynamics of a system, which is an extension of the Hamiltonian derived in the mean field approximation in Chapter 3. We showed that within the mean field approximation the interwell particle imbalance and the relative phase difference between the condensates are suitable choice of variables to describe the coherent particle tunneling between the condensates. Based on this fact and, in order to be able to compare our results with the results obtained in this approximation, we introduced the intrawell particle imbalance normalized to the total particle number and kept the definition of the relative phase difference given in the mean field regime.

We presented at the end of chapter 2 the collisionless regime for a system of bosons confined in a trap in the presence of a condensate fraction regime, which is described within the Bogoliubov-Hartree-Fock approximation. This approximation only displays the density fluctuations. Similarly, we showed in Section 4.2.1 the Bogoliubov-Hartree-Fock equation and their associated equations of motion for both condensates modes, which we analyzed numerically in Chapter 5.

The numerical results in Chapter 5 display a transition from the dynamics governed by the mean field approximation and the dynamics dominated by the am-

plitude oscillations of the condensates. This transition appears to be very strong correlated with the strength of the auxiliary bypass, as seen from the displacement of particles from one condensate to the other over the excited states.

Thermalization effects are not included in the collisionless regime (within the Hartree-Fock approximation). Future work demands the inclusion of the second and higher order terms in order to visualize thermalization. Part of this analytical work is presented in Appendix A. Extra effort is still needed to simplify the collision terms appearing on the right-hand side of the kinetic equations.

Appendix A

Analytical Continuation Procedure

We have seen in chapter 2.1 that the convolution of the selfenergy and the full propagator appears on the right hand side of the equation of motion for the Green's function, where the time integration is along the closed time path, depicted in Fig. (2.2), and the integration of the space is the usual one.¹ We are therefore now interested in how to formulate such quantities in real time.

We consider, for instance,

$$C(\tau_1, \tau_{1'}) = \int_c d\tau A(\tau_1, \tau)B(\tau, \tau_{1'}), \quad (\text{A.0.1})$$

where A and B are functions of the contour variable, and the involved contour is the closed time path depicted in Fig. 2.1.

Now we demonstrate the analytical continuation procedure for the case of $C^<$. The *lesser* quantity means, that the contour time τ_1 appears earlier than the contour time $\tau_{1'}$. Now making use of analyticity we deform the contour c in two the contour $c_1 + c_{1'}$, shown in Fig. A.1. The expression in Eq. (A.0.1), for the chosen contour ordering, therefore becomes

$$\begin{aligned} C^<(\tau_1, \tau_{1'}) &= \int_{c_1} d\tau A(\tau_1, \tau)B(\tau, \tau_{1'}) + \int_{c_{1'}} d\tau A(\tau_1, \tau)B(\tau, \tau_{1'}) \\ &= \int_{c_1} d\tau A(\tau_1, \tau)B^<(\tau, \tau_{1'}) + \int_{c_{1'}} d\tau A^>(\tau_1, \tau)B(\tau, \tau_{1'}), \end{aligned} \quad (\text{A.0.2})$$

where we use in the last equality, that on the contour c_1 , $\tau_{1'}$ appears after $\tau \in c_1$ and on the contour $c_{1'}$, τ_1 appears before $\tau \in c_{1'}$.

¹For the finite temperature case, we got additional to the contour shown in Fig. (2.1), an appendix contour in the imaginary direction from t_0 to $t_0 - i\beta$, where β is the Boltzmann factor proportional to the inverse of the temperature. If we are not interested in the initial correlations we can let $t_0 \rightarrow -\infty$ and obtain the Keldysh contour, shown in Fig. (2.2)

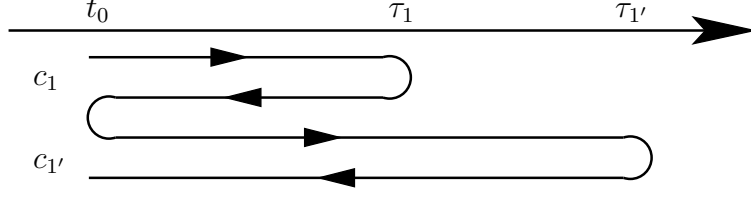


Figure A.1: Deforming the contour c into the contour built by the contours c_1 and $c_{1'}$.

Splitting in forward and backward contour parts we have

$$\begin{aligned}
C^<(\tau_1, \tau_{1'}) &= \int_{\vec{c}_1} d\tau A^>(\tau_1, \tau) B^<(\tau, \tau_{1'}) & \vec{c}_1 : \tau <_c \tau_1 \\
&= \int_{\overleftarrow{c}_1} d\tau A^<(\tau_1, \tau) B^<(\tau, \tau_{1'}) & \overleftarrow{c}_1 : \tau >_c \tau_1 \\
&= \int_{\vec{c}_1} d\tau A^<(\tau_1, \tau) B^<(\tau, \tau_{1'}) & \vec{c}_1 : \tau <_c \tau_{1'} \\
&= \int_{\overleftarrow{c}_1} d\tau A^<(\tau_1, \tau) B^>(\tau, \tau_{1'}) & \overleftarrow{c}_1 : \tau >_c \tau_{1'} .
\end{aligned} \tag{A.0.3}$$

Parameterizing the forward and backward contours according to

$$\tau(t) = t \quad t \in \{t_0, t_{1(\prime)}\}, \tag{A.0.4}$$

we get

$$\begin{aligned}
C^<(\tau_1, \tau_{1'}) &= \int_{-\infty}^{t_1} d\tau [A^>(\tau_1, \tau) - A^<(\tau_1, \tau)] B^<(\tau, \tau_{1'}) \\
&\quad - \int_{-\infty}^{t_{1'}} d\tau A^<(\tau_1, \tau) [B^>(\tau_1, \tau) - B^<(\tau_1, \tau)]
\end{aligned} \tag{A.0.5}$$

for $t_0 \longrightarrow -\infty$. Now introducing the retarded function

$$A^{\text{ret}}(\tau_1, \tau_{1'}) = \theta(\tau_1 - \tau_{1'}) [A^>(\tau_1, \tau) - A^<(\tau_1, \tau)] \tag{A.0.6}$$

and the advanced function

$$A^{\text{adv}}(\tau_1, \tau_{1'}) = -\theta(\tau_{1'} - \tau_1) [A^>(\tau_1, \tau) - A^<(\tau_1, \tau)] \tag{A.0.7}$$

we have the real time rule

$$C^<(\tau_1, \tau_{1'}) = \int_{-\infty}^{\infty} d\tau (A^{\text{ret}}(\tau_1, \tau) B^<(\tau, \tau_{1'}) + A^>(\tau_1, \tau) A^{\text{adv}}(\tau, \tau_{1'})) . \tag{A.0.8}$$

Analogously one can show

$$\begin{aligned}
C^>(\tau_1, \tau_{1'}) &= \int_{-\infty}^{\tau_1} d\tau [A^>(\tau_1, \tau) - A^<(\tau_1, \tau)] B^>(\tau, \tau_{1'}) \\
&\quad - \int_{-\infty}^{\tau_{1'}} d\tau A^>(\tau_1, \tau) [B^>(\tau_1, \tau) - B^<(\tau_1, \tau)] \quad (\text{A.0.9})
\end{aligned}$$

and

$$C^>(\tau_1, \tau_{1'}) = \int_{-\infty}^{\infty} d\tau (A^{\text{ret}}(\tau_1, \tau) B^>(\tau, \tau_{1'}) + A^<(\tau_1, \tau) A^{\text{adv}}(\tau, \tau_{1'})) . \quad (\text{A.0.10})$$

Appendix B

Inclusion of Collisions

In chapter 2 we presented the non-equilibrium extension of the field theory for a system of bosons in the presence of a condensate. Within this frame we derived the quantum Boltzmann equation for the particle outside of the condensate and the associated equation of motion for the condensate wave function. The right hand side of the equations display collision integrals containing processes of second and higher order that are responsible for thermalization effects. In chapter 4 we presented the model in consideration and derived in section 4.2 its corresponding Boltzmann equations for the distribution functions and its associated equations for the two condensate wave functions. These Boltzmann equations also presented the second and higher order processes in collisions integrals on the right hand side of the equations. Now we want to analyze these integrals under the assumption, that the selfenergies and the Green's functions are slowly varying functions for the center of mass time T , i.e. that they are sharply peaked around $\tau = 0$.

We consider

$$\begin{aligned}
 & \int_{-\infty}^t d\bar{t} \mathbf{\Gamma}_{nl}(t, \bar{t}) \mathbf{G}'_{lm}(\bar{t}, t') \\
 &= \int_{-\infty}^{T+\tau/2} d\bar{t} \mathbf{\Gamma}_{nl} \left(\frac{\bar{t} + T + \tau/2}{2}, T + \tau/2 - \bar{t} \right) \mathbf{G}'_{lm} \left(\frac{\bar{t} + T - \tau/2}{2}, \bar{t} - T + \tau/2 \right) \\
 &= \int_{-\infty}^{\tau} d\bar{t} \mathbf{\Gamma}_{nl} (T + \bar{t}/2, \tau - \bar{t}) \mathbf{G}'_{lm} (T + \bar{t}/2 - \tau/2, \bar{t}), \tag{B.0.1}
 \end{aligned}$$

where $\mathbf{\Gamma}_{nl}$ is the "spectral weight" of the selfenergy matrix containing second and higher order processes and \mathbf{G}' is the non-condensate matrix Green's function introduced in Eq. (4.2.3). Because the selfenergies involved in $\mathbf{\Gamma}_{nl}$, the Green's function are to be sharply peaked about $\tau = 0$, we can neglect the necessarily small quantities

added to T in Eq. (B.0.1). Then we get in first order

$$\begin{aligned} \int_{-\infty}^t d\bar{t} \mathbf{\Gamma}_{nl}(t, \bar{t}) \mathbf{G}'_{lm}{}^{<}(\bar{t}, t') &\approx \int_{-\infty}^{\tau} d\bar{t} \mathbf{\Gamma}_{nl}(T, \tau - \bar{t}) \mathbf{G}'_{lm}{}^{<}(T, \bar{t}) \\ &\approx \frac{1}{2} \int_{-\infty}^{\infty} d\bar{t} \mathbf{\Gamma}_{nl}(T, \tau - \bar{t}) \mathbf{G}'_{lm}{}^{<}(T, \bar{t}). \end{aligned} \quad (\text{B.0.2})$$

The other term on the right hand side of Eq. (4.2.3) contains the integral

$$\begin{aligned} &\int_{-\infty}^{t'} d\bar{t} (\mathbf{\Sigma}_c)_{nl}{}^{<}(t, \bar{t}) \mathbf{A}_{lm}(\bar{t}, t') \\ &= \int_{-\infty}^{T-\tau/2} d\bar{t} (\mathbf{\Sigma}_c)_{nl}{}^{<} \left(\frac{\bar{t} + T + \tau/2}{2}, T + \tau/2 - \bar{t} \right) \mathbf{A}_{lm} \left(\frac{\bar{t} + T - \tau/2}{2}, \bar{t} - T + \tau/2 \right) \\ &= \int_{\tau}^{\infty} d\bar{t} (\mathbf{\Sigma}_c)_{nl}{}^{<} (T - \bar{t}/2 + \tau/2, \bar{t}) \mathbf{A}_{lm}(T - \bar{t}/2, \tau - \bar{t}), \end{aligned} \quad (\text{B.0.3})$$

involving the collision selfenergy $(\mathbf{\Sigma}_c)_{nl}{}^{<}$ and the spectral weight function \mathbf{A} corresponding to the non-condensate particle Green's function. We again make use of the fact that only small values of τ and \bar{t} are important. Thus we get

$$\begin{aligned} \int_{-\infty}^{t'} d\bar{t} (\mathbf{\Sigma}_c)_{nl}{}^{<}(t, \bar{t}) \mathbf{A}_{lm}(\bar{t}, t') &\approx \int_{\tau}^{\infty} d\bar{t} (\mathbf{\Sigma}_c)_{nl}{}^{<}(T, \bar{t}) \mathbf{A}_{lm}(T, \tau - \bar{t}) \\ &\approx \frac{1}{2} \int_{-\infty}^{\infty} d\bar{t} (\mathbf{\Sigma}_c)_{nl}{}^{<}(T, \bar{t}) \mathbf{A}_{lm}(T, \tau - \bar{t}). \end{aligned} \quad (\text{B.0.4})$$

Now inserting Eq. (B.0.2) and Eq. (B.0.2) in the real time Dyson-Belayev equation (4.2.3) we obtain

$$\begin{aligned} &\sum_l \int d\bar{t} [(\mathbf{G}_0)_{nl}^{-1}(t, \bar{t}) - \mathbf{\Sigma}_{nl}^{HF}(t, \bar{t})] \mathbf{G}'_{lm}{}^{\geq}(\bar{t}, t') \\ &= \frac{1}{2} \sum_l \int_{-\infty}^{\infty} d\bar{t} [(\mathbf{\Sigma}_c)_{nl}{}^{>}(T, \tau - \bar{t}) \mathbf{G}'_{lm}{}^{<}(T, \bar{t}) - (\mathbf{\Sigma}_c)_{nl}{}^{<}(T, \tau - \bar{t}) \mathbf{G}'_{lm}{}^{>}(T, \bar{t})]. \end{aligned} \quad (\text{B.0.5})$$

Having the collision integrals in this form, it is easier to deal with them. For example, in order to compute the Fourier transform of Eq. (B.0.5) is enough to make use of the convolution theorem.

Replacing the functions $(I_{nm}^<)_{11}$ (and $(I_{nm}^<)_{12}$) in the Eq. (4.2.20) and Eq. (4.2.21) by the 11 (12) component of the difference (sum) of the Fourier transform of the right hand side of Eq. (B.0.5) and its hermitian conjugated we get.

$$\begin{aligned}
& \left[\frac{\partial}{\partial T} + i(\epsilon_n - \epsilon_m) + \frac{\partial}{\partial T} \Sigma_{11}^{HF}(T) \frac{\partial}{\partial \omega} \right] G_{nm}^<(\omega, T) \\
&= -i \left[\Sigma_{12}^{HF}(T) F_{nm}^{\dagger <}(\omega, T) - \Sigma_{21}^{HF}(T) F_{nm}^<(\omega, T) \right] \\
&\quad - \frac{1}{2} \left[\frac{\partial \Sigma_{12}^{HF}(T)}{\partial T} \frac{\partial}{\partial \omega} F_{nm}^{\dagger <}(\omega, T) + \frac{\partial \Sigma_{21}^{HF}(T)}{\partial T} \frac{\partial}{\partial \omega} F_{nm}^<(\omega, T) \right] \\
&\quad - \frac{i}{2} \sum_l \left[\{ (\Sigma_{nl}^{c>})_{11}(\omega, T) G_{lm}^<(\omega, T) - (\Sigma_{nl}^{c<})_{11}(\omega, T) G_{lm}^>(\omega, T) \} + \right. \\
&\quad + \{ G_{nl}^<(\omega, T) (\Sigma_{lm}^{c>})_{11}(\omega, T) - G_{nl}^>(\omega, T) (\Sigma_{lm}^{c<})_{11}(\omega, T) \} + \\
&\quad + \left. \{ (\Sigma_{nl}^{c>})_{12}(\omega, T) F_{lm}^{\dagger <}(\omega, T) - (\Sigma_{nl}^{c<})_{12}(\omega, T) F_{lm}^{\dagger >}(\omega, T) \} + \right. \\
&\quad + \left. \{ F_{nl}^<(\omega, T) (\Sigma_{lm}^{c>})_{21}(\omega, T) - F_{nl}^>(\omega, T) (\Sigma_{lm}^{c<})_{21}(\omega, T) \} \right], \tag{B.0.6}
\end{aligned}$$

$$\begin{aligned}
& \left[\frac{\partial}{\partial T} + i(\epsilon_n + \epsilon_m) - i2\mu + i2\Sigma_{11}^{HF}(T) \right] F_{nm}^<(\omega, T) \\
&= -i\Sigma_{12}^{HF}(T) \left[\bar{G}_{nm}^<(\omega, T) + G_{nm}^<(\omega, T) \right] \\
&\quad - \frac{1}{2} \frac{\partial \Sigma_{12}^{HF}(T)}{\partial T} \frac{\partial}{\partial \omega} \left[\bar{G}_{nm}^<(\omega, T) - G_{nm}^<(\omega, T) \right] \\
&\quad - \frac{i}{2} \sum_l \left[\left\{ (\Sigma_{nl}^{c>})_{12}(\omega, T) \bar{G}_{lm}^<(\omega, T) - (\Sigma_{nl}^{c<})_{12}(\omega, T) \bar{G}_{lm}^>(\omega, T) \right\} - \right. \\
&\quad - \{ G_{nl}^<(\omega, T) (\Sigma_{lm}^{c>})_{12}(\omega, T) - G_{nl}^>(\omega, T) (\Sigma_{lm}^{c<})_{12}(\omega, T) \} + \\
&\quad + \left. \{ (\Sigma_{nl}^{c>})_{11}(\omega, T) F_{lm}^<(\omega, T) - (\Sigma_{nl}^{c<})_{11}(\omega, T) F_{lm}^>(\omega, T) \} - \right. \\
&\quad - \left. \{ F_{nl}^<(\omega, T) (\Sigma_{lm}^{c>})_{22}(\omega, T) - F_{nl}^>(\omega, T) (\Sigma_{lm}^{c<})_{22}(\omega, T) \} \right]. \tag{B.0.7}
\end{aligned}$$

These two equations are the kinetic equations for the components of $\mathbf{G}'_{nm}^<$ introduced in section 4.2. They display collisions under the assumption that the selfenergies and the Green's functions are sharply peaked about $\tau = 0$. The generalization of the discussion of the collision terms is presented elsewhere [36, 41].

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